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ON CERTAIN SELF-RECIPROCAL FUNCTIONS

By

S. C. Mitra

(*Received January 10, 1949*)

1. Ramanujan (1915) has proved that

$$\sqrt\beta.[F_c(\beta)-F_c(3\beta)-F_c(5\beta)+\cdots] = \sqrt\alpha.[f(\alpha)-f(3\alpha)-f(5\alpha)+\cdots]\,; \quad \alpha\beta=\pi/4$$

and

$$F_c(u) = \sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty f(t)\cos utdt.$$

The theorem holds good when (1) $f(x)$ is of bounded variation in $(0, \infty)$, and tends to 0 as $x \to \infty$; (2) if also

$$\int_0^{\to\infty} f(x)dx$$

exists and (3) if also $f(x)$ is continuous (Titchmarsh, 1937, p. 62).

If $\alpha = 1$, then $\beta = \pi/4$. Therefore

$$\tfrac{1}{2}\sqrt\pi.[F_c(\pi/4)-F_c(3\pi/4)-F_c(5\pi/4)+\cdots] = [f(1)-f(3)-f(5)+\cdots].$$

Let $f(t) = (\frac{1}{2}tz)^\nu K_\nu(tz)$. Since (Watson, 1931)

$$\int_0^\infty (\tfrac{1}{2}tz)^\nu K_\nu(tz)\cos\{\tfrac{1}{4}(2n-1)\pi t\}dt = \frac{\Gamma(\frac{1}{2})\Gamma(\nu+\frac{1}{2})z^{2\nu}}{2\{z^2+(2n-1)^2\frac{1}{16}\pi^2\}^{\nu+\frac{1}{2}}};$$

we have

$$(\tfrac{1}{2}z)^\nu K_\nu(z)-(\tfrac{3}{2}z)^\nu K_\nu(3z)-(\tfrac{5}{2}z)^\nu K_\nu(5z)+\cdots$$

$$= \tfrac{1}{2}\sqrt(\tfrac{1}{2}\pi).\Gamma(\nu+\tfrac{1}{2})z^{2\nu}\left\{\frac{1}{(z^2+\frac{1}{16}\pi^2)^{\nu+\frac{1}{2}}}-\frac{1}{(z^2+\frac{9}{16}\pi^2)^{\nu+\frac{1}{2}}}-\frac{1}{(z^2+\frac{25}{16}\pi^2)^{\nu+\frac{1}{2}}}+\cdots\right\}.$$

We know that (Watson, 1922)

$$\int_0^\infty \frac{y^{\mu+1}J_\mu(xy)}{(y^2+k^2)^{\nu+\frac{1}{2}}}dy = \frac{(\frac{1}{2}x)^{\nu-\frac{1}{2}}k^{\mu-\nu+\frac{1}{2}}}{\Gamma(\nu+\frac{1}{2})}K_{\mu-\nu+\frac{1}{2}}(kx).$$

Let

$$F_\nu(az) = (\tfrac{1}{2}az)^\nu K_\nu(az)-(\tfrac{3}{2}az)^\nu K_\nu(3az)-(\tfrac{5}{2}az)^\nu K_\nu(5az)+\cdots$$

$$= \tfrac{1}{2}\sqrt(\tfrac{1}{2}\pi).\Gamma(\nu+\tfrac{1}{2})z^{2\nu}a^{2\nu}\left\{\frac{1}{(a^2z^2+\frac{1}{16}\pi^2)^{\nu+\frac{1}{2}}}-\frac{1}{(a^2z^2+\frac{9}{16}\pi^2)^{\nu+\frac{1}{2}}}-\frac{1}{(a^2z^2+\frac{25}{16}\pi^2)^{\nu+\frac{1}{2}}}+\cdots\right\}.$$

Hence

$$\int_0^\infty \sqrt(xy).J_\mu(xy)(ay)^{\mu-2\nu+\frac{1}{2}}F_\nu(ay)dy = \frac{\sqrt\pi}{2a}\left(\frac{x}{2a}\right)^{2\nu-\mu-\frac{1}{2}}F_{\mu-\nu+\frac{1}{2}}\left(\frac{\pi x}{4a}\right).$$

Let $a = \frac{1}{2}\sqrt{\pi}$, $\mu = 2\nu - \frac{1}{2}$. Hence

$$\int_0^\infty \sqrt{(xy)}.J_{2\nu-\frac{1}{2}}(xy)F_\nu(\tfrac{1}{2}\sqrt{\pi}.y)dy = F_\nu(\tfrac{1}{2}\sqrt{\pi}\, x)$$

Hence $F_\nu(\frac{1}{2}\sqrt{\pi}\, x)$ is self-reciprocal in the Hankel transform of order $2\nu - \frac{1}{2}$

To justify the change in the order of integration and summation, we assume $\nu > 0$ and $-1 < \mu < 2\nu - 2$

Let us consider the integral

$$\int_0^\infty \frac{J_\mu(xy)y^{\mu+1}dy}{\{(2n-1)^2\frac{1}{16}\pi^2 + y^2\}^{\nu+\frac{1}{2}}}.$$

We can divide the range of integration into two parts, from zero to one and from one to infinity. Then

$$\left|\int_0^1 \frac{J_\mu(xy)y^{\mu+1}dy}{\{(2n-1)^2\frac{1}{16}\pi^2 + y^2\}^{\nu+\frac{1}{2}}}\right| < \frac{1}{\{(2n-1)\frac{1}{4}\pi\}^{2\nu+1}}\int_0^1 |J_\mu(xy)y^{\mu+1}|\, dy\,;\ (y \leq 1)$$

Now

$$\int_0^1 |J_\mu(xy)y^{\mu+1}| \text{ is convergent and } \sum \frac{1}{\{(2n-1)\frac{1}{4}\pi\}^{2\nu+1}}$$

is convergent when $2\nu + 1 > 1$, *i.e.* when $\nu > 0$. When $y > 1$,

$$\left|\int_1^\infty \frac{J_\mu(xy)y^{\mu+1}dy}{\{(2n-1)^2\frac{1}{16}\pi^2 + y^2\}^{\nu+\frac{1}{2}}}\right| = \left|\int_1^\infty \frac{J_\mu(xy)y^{\mu+1}dy}{\{(2n-1)^2\frac{1}{16}\pi^2 + y^2\}^{\frac{1}{2}\mu+1}\,\{\frac{1}{16}(2n-1)^2\pi^2 + y^2\}^{\nu-\frac{1}{2}\mu-\frac{1}{2}}}\right|$$

$$< \int_1^\infty \frac{|y^{-1}J_\mu(xy)|\, dy}{\{(2n-1)\frac{1}{4}\pi\}^{2\nu-\mu-\frac{1}{2}}}.$$

We note that

$$\int_1^\infty |y^{-1}J_\mu(xy)|\, dy \text{ and } \sum_{n=1}^\infty \frac{1}{\{(2n-1)\frac{1}{4}\pi\}^{2\nu-\mu-\frac{1}{2}}}$$

are convergent It follows from these two inequalities that the changes in the order of summation and integration between the ranges from zero to one and one to infinity are justifiable.

Combining the two results, we get

$$\int_0^\infty \sqrt{(xy)}.J_\mu(xy)(ay)^{\mu-2\nu+\frac{1}{2}}F_\nu(ay)dy = \frac{\sqrt{\pi}}{2^{\nu+1}}x^\nu a^{\mu-2\nu-\frac{1}{2}}\left[\left(\frac{\pi}{4a}\right)^{\mu-\nu+\frac{1}{2}}K_{\mu-\nu+\frac{1}{2}}\left(\frac{\pi x}{4a}\right)\right.$$

$$\left.-\left(\frac{3\pi}{4a}\right)^{\mu-\nu+\frac{1}{2}}K_{\mu-\nu+\frac{1}{2}}\left(\frac{3\pi x}{4a}\right) - \left(\frac{5\pi}{4a}\right)^{\mu-\nu+\frac{1}{2}}K_{\mu-\nu+\frac{1}{2}}\left(\frac{5\pi x}{4a}\right) + \cdots\right].$$

Now the series on the right is an analytic function of both μ and ν in any finite domains of values of μ and ν; and by dividing the range of integration on the left into the ranges (0, 1) and (1, ∞) and using the two expressions in the respective ranges, we see that each of the two integrals is an analytic function of both μ and ν in the domains given by

$$-1 < \mu \leq 2\nu - \tfrac{1}{2} \text{ and } \nu \geq 0.$$

It is unnecessary to suppose ν to be a real number. If ν is a complex number, we shall take $\mathcal{R}(\nu)$ instead of ν.

2. Let $f(t) = K_0(tz)$ Then $F_c(x) = \sqrt{(\frac{1}{2}\pi)}/\sqrt{(x^2+z^2)}$. We obtain as before

$$K_0(\tfrac{1}{2}\sqrt{\pi}.x) - K_0(\tfrac{3}{2}\sqrt{\pi}.x) - K_0(\tfrac{5}{2}\sqrt{\pi}.x) + \cdots$$

$$= \sqrt{\left(\frac{\pi}{2}\right)}\left\{\frac{1}{\sqrt{(x^2+\frac{1}{4}\pi)}} - \frac{1}{\sqrt{(x^2+\frac{9}{4}\pi)}} - \frac{1}{\sqrt{(x^2+\frac{25}{4}\pi)}} + \cdots\right\}$$

Now put $x = p$ We know from Laplace transform (Carslaw & Jaeger, 1941) that

$$\frac{p}{\sqrt{(p^2+a^2)}} \doteqdot J_0(at) \text{ and } pK_0(ap) = \frac{1}{\sqrt{(t^2-a^2)}}, \quad (t > a) \text{ and } 0, \ (0 < t < a).$$

Then since

$$pK_0\left(\frac{2n-1}{2}\sqrt{\pi}.p\right) = \frac{1}{\sqrt{\{t^2-(n-\frac{1}{2})^2\pi\}}}; \quad (t > (n-\tfrac{1}{2})\sqrt{\pi}),$$

we have on interpretation,*

$$\left\{\frac{1}{\sqrt{(t^2-\frac{1}{4}\pi)}} - \frac{1}{\sqrt{(t^2-\frac{9}{4}\pi)}} - \frac{1}{\sqrt{(t^2-\frac{25}{4}\pi)}} + \cdots\right\} = \sqrt{\left(\frac{\pi}{2}\right)} \cdot (t) \qquad (1)$$

where

$$F(t) = J_0(\tfrac{1}{2}\sqrt{\pi}.t) - J_0(\tfrac{3}{2}\sqrt{\pi}.t) - J_0(\tfrac{5}{2}\sqrt{\pi}.t) + J_0(\tfrac{7}{2}\sqrt{\pi}.t) + \cdots$$

We shall now show that $F(t)$ is self-reciprocal in the sine-transform.

Since both the series are uniformly convergent, we can multiply $F(t)$ by $\sin kt$ and integrate term by term On integration, we get, on making use of (1),

$$\sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty F(t) \sin kt dt = F(k).$$

3 We have the result (Titchmarsh, 1937, p 202)

$$\int_0^\infty \frac{J_0(xt)}{\sqrt{(1+t^2)}} dt = I_0(x)K_0(x)$$

Therefore

$$\int_0^\infty \frac{F(xt)}{\sqrt{(1+t^2)}} dt = \int_0^\infty \left\{J_0(\tfrac{1}{2}\sqrt{\pi}.xt) - J_0(\tfrac{3}{2}\sqrt{\pi}.xt) - J_0(\tfrac{5}{2}\sqrt{\pi}.xt) + \cdots\right\} \frac{dt}{\sqrt{(1+t^2)}}$$

$$= I(\tfrac{1}{2}\sqrt{\pi}\ x)K_0(\tfrac{1}{2}\sqrt{\pi}.x) - I_0(\tfrac{3}{2}\sqrt{\pi}\ x)K_0(\tfrac{3}{2}\sqrt{\pi}.x) - I_0(\tfrac{5}{2}\sqrt{\pi}\ x)K_0(\tfrac{5}{2}\sqrt{\pi}.x) + \cdots$$

Since $F(t)$ is self reciprocal in the sine transform, we have by multiplying by $\sin kx$ and integrating with respect to x

$$\int_0^\infty \frac{dt}{\sqrt{(1+t^2)}} \int_0^\infty F(xt) \sin kx dx = \int_0^\infty \{I_0(\tfrac{1}{2}\sqrt{\pi}.x)K_0(\tfrac{1}{2}\sqrt{\pi}.x)$$

$$- I_0(\tfrac{3}{2}\sqrt{\pi}\ x)K_0(\tfrac{3}{2}\sqrt{\pi}.x) - I_0(\tfrac{5}{2}\sqrt{\pi}.x)K_0(\tfrac{5}{2}\sqrt{\pi}\ x) + \cdots\} \sin kx dx$$

---

* (1) Can be deduced from Ramanujan's formula also.

Left hand side is

$$\int_0^\infty \frac{dt}{\sqrt{(1+t^2)}}\cdot\frac{1}{t}\int_0^\infty F(x)\sin\frac{k}{t}x\,dx$$

$$=\sqrt{\left(\frac{\pi}{2}\right)}\int_0^\infty \frac{1}{t\sqrt{(1+t^2)}}F\left(\frac{k}{t}\right)dt=\sqrt{\left(\frac{\pi}{2}\right)}\int_0^\infty \frac{F(kt)}{\sqrt{(1+t^2)}}\,dt.$$

On integrating term by term, we find that the last integral is equal to

$$\sqrt{(\pi/2)}.\{I_0(\tfrac{1}{2}\sqrt{\pi}.x)K_0(\tfrac{1}{2}\sqrt{\pi}.x)-I_0(\tfrac{3}{2}(\sqrt{\pi}.x)K_0(\tfrac{3}{2}\sqrt{\pi}.x)-I_0(\tfrac{5}{2}\sqrt{\pi}.x)K_0(\tfrac{5}{2}\sqrt{\pi}.x)+\cdots\}.$$

We have now to justify the change in the order of integration and term by term integration in the last integral. $F(t)$ being uniformly convergent, term by term integration is easily justifiable.

We shall now show that

$$\int_0^\infty \frac{dt}{\sqrt{(t^2+1)}}\int_0^\infty F(xt)\sin kx\,dx=\int_0^\infty \sin kx\,dx\int_0^\infty \frac{F(xt)dt}{\sqrt{(t^2+1)}}.$$

The right hand side can be written as

$$\int_0^\infty \sin kx\,dx\int_0^\infty \frac{F(t)dt}{\sqrt{(t^2+x^2)}}.$$

We note (Hobson, 1907) that $F(t)$ is a continuous function of t, and (1) the repeated integrals between finite limits always exist and are equal,

(2) $\displaystyle\int_0^\infty \frac{F(xt)}{\sqrt{(t^2+1)}}\sin kx\,dx$ is uniformly convergent in an arbitrary interval,

(3) $\displaystyle\int_0^\infty \frac{F(t)}{\sqrt{(t^2+x^2)}}\sin kx\,dt$ satisfies the same condition,

and

(4) $\displaystyle\int_0^\infty dt\int_0^X \frac{F(xt)\sin kx}{\sqrt{(t^2+1)}}\,dx$ converges uniformly in an unlimited interval.

The last condition can also be replaced by the condition

(5) $\displaystyle\int_0^\infty dx\int_0^T \frac{F(xt)\sin kx}{\sqrt{(t^2+1)}}\,dt$ which also converges in an unlimited interval.

Therefore change in the order of integration is justifiable.

4. The theorem for the sine-transform (Titchmarsh, 1937, p. 66) is

$$\sqrt{\beta}.\{F_s(\beta)-F_s(3\beta)+\cdots\}=\sqrt{\alpha}.\{f(\alpha)-f(3\alpha)+f(5\alpha)-\cdots\},\ \text{where } \alpha\beta=\tfrac{1}{2}\pi.$$

$$F_s(x)=\sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty f(t)\sin xt\,dt$$

Let $\alpha=1$, then $\beta=\frac{1}{2}\pi$ and $f(t)=(\frac{1}{2}tz)^{\nu+1}K_\nu(tz)$.

Since

$$\int_0^\infty t^{\nu+1}K_\nu(tz)\sin at\,dt = (2z)^\nu\frac{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2}+\nu)a}{(a^2+z^2)^{\nu+3/2}}$$

we get after slight reductions,

$$\sum_{n=1}^\infty(-1)^{n-1}(\overline{n-\tfrac{1}{2}}z)^{\nu+1}K_\nu(\overline{2n-1}z) = \sum_{n=1}^\infty\frac{(-1)^{n-1}\pi\sqrt{\pi}.(2n-1)z^{2\nu+1}\Gamma(\frac{3}{2}+\nu)}{4\{(2n-1)^2\pi^2/4+z^2\}^{\nu+3/2}}. \tag{2}$$

Hence, putting

$$F_\nu(z) = \sum_{n=1}^\infty(-1)^{n-1}(\overline{n-\tfrac{1}{2}}z)^{\nu+1}K_\nu(\overline{2n-1}z),$$

we have

$$\int_0^\infty\sqrt{(xy)}.J_\mu(xy)(ay)^{\mu-2\nu-\frac{1}{2}}F_\nu(ay)dy$$

$$= \sum_{n=1}^\infty\frac{(-1)^{n-1}\pi^{\mu-\nu+1}(2n-1)^{\mu-\nu+\frac{1}{2}}(ax)^{\nu+1}K_{\mu-\nu-\frac{1}{2}}(\overline{n-\frac{1}{2}}\,\pi x/a)}{2^{\mu+2}a^{2\nu+3}}$$

Let $a = \sqrt{(\frac{1}{2}\pi)}$, $\mu = 2\nu+\frac{1}{2}$ We get

$$F_\nu\left(x\sqrt{\frac{\pi}{2}}\right) = \sum_{n=1}^\infty(-1)^{n-1}\left(\overline{n-\tfrac{1}{2}}\sqrt{\frac{\pi}{2}}\cdot x\right)^{\nu+1}K_\nu\left(\overline{2n-1}\sqrt{\frac{\pi}{2}}\cdot x\right)$$

is self-reciprocal in the Hankel transform of order $2\nu+\frac{1}{2}$. The change in the order of integration and summation is justifiable as before and the range of validity is given by $-1<\mu<2\nu+\frac{3}{2}$.

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GAUSS-POINTS IN n-DIMENSIONAL SPACE

By

Sahib Ram Mandan

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A Gauss-point w. r. t. two simplexes in a space of n dimensions is one which has the same co-ordinates in some order-refered to either simplex, if the system of co-ordinates is such that the centroid of the simplex of reference is the unit-point.

There are in all $(n+1)!$ *Gauss-points.*[*] *They all lie on a quadratic-variety* W *and by* $(n-r+1)!$ *in*

$$\frac{(n+1)!\,{}^{n+1}C_r}{(n-r+1)!}, \quad i.e., \quad \frac{\{(n+1)!\}^2}{\{(n-r+1)!\}^2 r!},$$

$(n-r)$*-flats* $(r = 1, 2, \ldots, n-1)$, *in particular by* $n!$ *in* $(n+1)^2$ *primes. These primes are faces of two sets of* $n+1$ *simplexes such that simplexes of one set are mutually perspective from the centroid of one of the given simplexes and the primes of perspectivity pass through the centroid and a* $(n-2)$*-flat of the 2nd given simplex. The two centroids constitute a pair of conjugate points for* W.

Let S_n be a space of n dimensions and S, S' be two simplexes therein. Let the the co-ordinates of a point P w. r. t. S be x_i $(i = 0, 1, 2, \ldots, n)$ and those w. r. t. S' be x_i'. Now as the centroid of the simplex of reference is the unit-point, we have $\sum x_i = 1 = \sum x_i'$, or $\sum x_i - \sum x_i' \equiv 0$. This identity shows that the $(n+1)!$ sets of $n+1$ primes of the type $x_i = x_j'$ concur at $(n+1)!$ points which satisfy the condition of a G pt. They again lie on the quadratic-variety W given by the equation[†] $\sum_{i,j=0}^{n} x_i x_j = \sum_{i,j=0}^{n} x_i' x_j'$, or $\sum x_i^2 = \sum x_i'^2$, where $i \neq j$, the two being equivalent on account of the above identity.

The $(n+1)!$ G. pts. may be put down as follows

$$x_0 = x_0',\ x_1 = x_1',\ \ldots,\ x_{r-1} = x_{r-1}',\ x_r = x_r',\ x_{r+1} = x_{r+1}',\ \ldots,\ x_{n-1} = x_{n-1}',\ x_n = x_n';$$

$$x_0 = x_0',\ x_1 = x_1',\ \ldots,\ x_{r-1} = x_{r-1}',\ x_r = x_{r+1}',\ x_{r+1} = x_{r+2}',\ \ldots,\ x_{n-1} = x_n',\ x_n = x_r';$$

$$x_0 = x_0',\ x_1 = x_1',\ \ldots,\ x_{r-1} = x_{r-1}',\ x_r = x_{r+2}',\ x_{r+1} = x_{r+3}',\ \ldots,\ x_{n-1} = x_r',\ x_n = x_{r+1}'.$$

. . . .

The r primes $x_i = x_i'$ $(i = 0, 1, \ldots, r-1)$ have a $(n-r)$-flat common, by permuting x_j' $(j = r, \ldots, n)$ in different ways we get $(n-r+1)!$ G. pts. in this $(n-r)$-flat. Similarly in any such $(n-r)$-flat there are $(n-r+1)!$ G. pts.

---

[*] In what follows Gauss-point will be written as 'G. pt.'

[†] This was suggested to me by Dr. F. C. Auluck.

Let $r = 1$, we get $n!$ G. pts. in the prime $x_0 = x_0'$ Similarly we shall have $n!$ different G. pts. in each of the other n primes $x_0 = x_i'(i = 1, \ldots, n)$ Hence there are $(n+1)!$ G. pts. in all lying by $(n!)^2$ in the $(n+1)^2$ primes $x_i = x_j'$. The above scheme therefore contains $(n+1)!$ rows, each row containg $n+1$ primes Any r primes in a row will fix a $(n-r)$-flat and we shall have ${}^{n+1}C_r (n-r)$-flats from a row. Thus there will be $\{(n+1)!\,{}^{n+1}C_r/(n-r+1)!\}$ $(n-r)$-flats each containing $(n-r+1)!$ G. pts., for each row will occur then $(n-r+1)!$ times.

The $(n+1)^2$ primes may be put down in a square of $n+1$ rows and $n+1$ columns as below:

$$\begin{array}{lll}
x_0 = x_0', \ldots, & x_0 = x_r', \ldots, & x_0 = x_n'; \\
\ldots, & \ldots, & \ldots; \\
x_k = x_0', \ldots, & x_k = x_r', \ldots, & x_k = x_n'; \\
\ldots, & \ldots, & \ldots; \\
x_n = x_0', \ldots, & x_n = x_r', \ldots, & x_n = x_n'.
\end{array}$$

The primes in a row or a column fix a simplex. Thus these primes are faces of two sets of $n+1$ simplexes as we read them by rows or columns. The simplexes of rows are at once seen to be mutually perspective from the point given by the equations $x_0' = x_1' = \cdots = x_r' = x_{r+1}' \ldots = x_n'$. This point is no other than the centroid of the simplex S'. The primes of perspectivity are $x_i = x_j$ which pass through the centroid of S and $(n-2)$-flats of S givenby $x_i = 0 = x_j$ respectively Similarly the simplexes of columns are mutually perspective from the centroid of S and their primes of perspectivity pass through the centroid and $(n-2)$-flats of S'.

Further let $x_i' = \sum_{j=0}^{n} l_{ij} x_j$ $(i = 0, 1, \ldots, n)$ where $\sum_{i=0}^{n} l_{ij} = 1$. The polar prime of the centroid of the simplex S w. r. t W is

$$\sum_{j=0}^{n} x_i = \sum_{i=0}^{n} \left[x_i' \sum_{j=0}^{n} \left(\frac{\partial x_i'}{\partial x_j} \right) \right]$$

where $(\partial x_i'/\partial x_j)$ stands for $\partial x_i'/\partial x_j$ when $x_j = 1$ This equation reduces to

$$\sum_{i=0}^{n} x_i = \sum_{i=0}^{n} \left[x_i' \sum_{j=0}^{n} l_{ij} \right].$$

In it put $x_j' = 1$ Both the sides of the equation are then equal to $\sum_{j=0}^{n} \sum_{i=0}^{n} l_{ij}$ which is again equal to $n+1$ because of the above relations in x_i' and x_j Hence the polar prime of the centroid of the simplex S contains the centroid of the simplex S', *i.e.*, the two centroids are conjugate for W.

As a particular case let us take $n = 2$ to illustrate what is said above in a plane: *There are six G pts. with regard to two triangles in a plane. They lie on a conic and lie by two on nine lines which are sides of two sets of three triangles such that triangles of one set are mutually perspective from the centroid of one of the two given triangles* *and the lines of perspectivity are the medians of the second triangle. The two centroids are conjugate for the conic.*

This theorem can be proved independently by the converse of Pascal's theorem on a hexagon. *The medians of the given triangles are six Pascal lines of six hexagons formed by the six G. pts., the centroids being a pair of Steiner's* conjugate points.*

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* Baker, H. F., Principles of Geometry Vol. 2, p 231.

A STUDY OF THE GENERALISED LAPLACE INTEGRAL—I*

By

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Laplace integral

$$\varphi(p) = p\int_0^\infty e^{-px}\,h(x)dx \tag{1}$$

has been studied both extensively and intensively. It has also been taken as the basis of a transform, called Laplace transform, which has been developed under the name of Symbolic Calculus or Operational Calculus.

At the Indian Science Congress, 1945, Dr. R. S. Varma (1947) gave a generalisation of the integral (1) in the form

$$\varphi(p) = p\int_0^\infty (2xp)^{-\frac{1}{4}}\,W_{k,m}(2px)h(x)dx \tag{2}$$

where $W_{k,m}$ denotes Whittaker function (Whittaker and Watson, 1927, p. 346), defined by †

$$W_{k,m}(x) = \frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-m-k)}x^{m+\frac{1}{2}}e^{-\frac{1}{2}x}\cdot{}_1F_1\left\{\begin{matrix}\frac{1}{2}+m-k\\ 2m+1\end{matrix}\,;\ x\right\}+\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)}x^{-m+\frac{1}{2}}e^{-\frac{1}{2}x}\cdot{}_1F_1\left\{\begin{matrix}\frac{1}{2}-m-k\\ -2m+1\end{matrix}\,;\ x\right\}.$$

He has recently given an inversion formula for the integral (2)

Following Dr. Varma, we shall call $\varphi(p)$ as the Whittaker transform of $h(x)$, and $h(x)$ the original of $\varphi(p)$. It is possible to develop the transform based on the generalised Laplace integral and the present paper is an attempt towards this

For the sake of convenience, we shall take

$$\varphi(p) \underset{m}{\overset{k}{=}} h(x)$$

as equivalent to the integral (2).

The Laplace transform comes as a particular case of this, when $k = \frac{1}{4}$, $m = \pm\frac{1}{4}$, on account of the identity,

$$(2xp)^{-\frac{1}{4}}\,W_{\frac{1}{4},\pm\frac{1}{4}}(2px) \equiv e^{-xp}.$$

As is conventional, the notation

$$\varphi(p) \underset{\cdot}{=} h(x)$$

will mean integral (1).

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* I am indebted to Dr. R. S. Varma, Reader in Mathematics, Lucknow University, under whose able guidance I have carried out these investigations.

† We shall have the occasion to use this definition frequently and we shall henceforth call it as (D. 1).

Other special cases of the general transform, as pointed out by Dr Varma, are the following:—

(1) When $k = 0$, we have

$$W_{0,m}(x) = \sqrt{(x/\pi)}.K_m(x/2),$$

the general transform reduces to

$$\varphi(p) = (p/\sqrt{\pi}) \int_0^\infty (2xp)^{\frac{1}{2}} K_m(px) h(x) dx$$

and will be known as K-transform and symbolically denoted by

$$\varphi(p) \overset{0}{\underset{m}{=}} h(x)$$

(2) When $k = \frac{1}{2} + \frac{1}{2}n + l$ and $m = \pm\frac{1}{2}n$, we have

$$W_{\frac{1}{2}+\frac{1}{2}n+l,\, \pm\frac{1}{2}n}(x) = (-1)^l e^{-\frac{1}{2}x} x^{\frac{1}{2}n+\frac{1}{2}} L_l^n(x),$$

l being a positive integer and the general transform gives

$$\varphi(p) = (-1)^l l!\, p \int_0^\infty (2xp)^{\frac{1}{2}n+\frac{1}{2}} e^{-px} L_l^n(2xp) h(x) dx$$

and will be known as L_l^n-transform, symbolically denoted by

$$\varphi(p) \overset{\frac{1}{2}+\frac{1}{2}n+l}{\underset{\pm\frac{1}{2}n}{=}} h(x).$$

(3) When $k = \frac{1}{2}n + \frac{1}{4}$, $m = \pm\frac{1}{4}$, we have

$$D_n(2\sqrt{x}) = 2^{\frac{1}{2}n}(2x)^{-\frac{1}{4}} W_{\frac{1}{2}n+\frac{1}{4},\, \pm\frac{1}{4}}(2x)$$

and the general transform gives

$$\varphi(p) = 2^{-\frac{1}{2}n} p \int_0^\infty D_n\{2\sqrt{(xp)}\} h(x) dx$$

and will be known as D_n-transform and symbolically denoted by

$$\varphi(p) \overset{\frac{1}{2}n+\frac{1}{4}}{\underset{\pm\frac{1}{4}}{=}} h(x)$$

Dr. Varma has given five theorems without proof. They are

Theorem 1 A. If

$$\varphi(p) \overset{k}{\underset{m}{=}} h(x),$$

then

$$\left(-p\frac{d}{dx}\right)^n \varphi(p) \overset{k}{\underset{m}{=}} \left(x\frac{d}{dx}\right)^n h(x).$$

Theorem 2 A. If

$$\varphi_1(p) \overset{k}{\underset{m}{=}} h_1(x)$$

and

$$\varphi_2(p) \overset{k}{\underset{m}{=}} h_2(x),$$

then
$$\int_0^\infty \varphi_1(x)h_2(x)\frac{dx}{x}=\int_0^\infty \varphi_2(x)h_1(x)\frac{dx}{x}.$$

Theorem 1 B. If
$$\varphi(p)\overset{k}{\underset{m}{=}}h(x),$$
then
$$\int_0^\infty \varphi(x)\frac{dx}{x}=\frac{\Gamma(5/4-m)\Gamma(5/4+m)}{2\Gamma(7/4-k)}\cdot{}_2F_1\left\{\begin{matrix}5/4+m,\ 5/4-m\\ 7/4-k\end{matrix};\ \tfrac{1}{2}\right\}\int_0^\infty h(x)\frac{dx}{x}.$$

Theorem 2 B. If
$$\varphi(p)\overset{k}{\underset{m}{=}}h(x)$$
and
$$h(p)\doteq\psi(x),$$
then
$$\varphi(p)=\frac{\Gamma(9/4+m)\Gamma(9/4-m)}{4p\Gamma(11/4-m)}\int_0^\infty {}_2F_1\left\{\begin{matrix}9/4+m,\ 9/4-m\\ 11/4-k\end{matrix};\ -\frac{s-p}{2p}\right\}\psi(s)\,ds$$

Theorem 1 C If
$$\varphi(p)\overset{k}{\underset{m}{=}}h(x),$$
then
$$\sum_{r=0}^{\infty}\frac{1}{r!}(1/\alpha-1)^r\varphi^{(r)}(p)=p\alpha^k\int_0^\infty(2xp)^{-\frac{1}{2}}e^{xp(1-\alpha)}W_{k,m}(2px\alpha)h(x)\,dx.$$

Theorems marked A have exactly the same form as the corresponding theorems in Operational Calculus, those marked B have generalised appearances, and whilst the one marked C has no analogue in the theory of Operational Calculus.

Among other generalisations of Laplace integral may be mentioned:

(1) The generalisation due to Meijer (1941),

(2) The generalisation due to Boas (1942).

The generalisations of Laplace integral given by Meijer (1941) and Varma are based on replacing e^{-px} in (1) by a kernel involving Whittaker Function, $W_{k,m}(x)$, which reduce to e^{-px} for certain values of the parameters k and m, whereas the generalisation due to Boas depends on replacing e^{-zt} by a kernal $g(z, t)$ which is in some sense "nearly" e^{-zt}.

I

1.1. We begin by giving a number of examples of the generalised Laplace transform based on the integral
$$\varphi(p)=p\int_0^\infty(2xp)^{-\frac{1}{2}}W_{k,m}(2px)h(x)\,dx$$

1.2 *Example 1.* When $h(x)=x^n$, we have* from an integral due to Goldstein (1932), *viz.*,

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* We shall refer this formula of Goldstein as (D 2).

$$\int_0^\infty x^{l-1} e^{-(\alpha^2+\frac{1}{2})x} W_{k,m}(x)\,dx = \frac{\Gamma(l+m+\frac{1}{2})\Gamma(l-m+\frac{1}{2})}{\Gamma(l-k+1)}\, {}_2F_1\left\{\begin{matrix} l+m+\frac{1}{2},\, l-m+\frac{1}{2}; \\ l-k+\frac{1}{2}; \end{matrix} -\alpha^2\right\},$$

$$l \pm m+\tfrac{1}{2} > 0 \text{ and } \mathbf{R}(\alpha^2+1) > 0,$$

$$\frac{\Gamma(n+m+5/4)\Gamma(n-m+5/4)}{2^{n+1}\Gamma(n-k+7/4)} p^{-n} . {}_2F_1\left\{\begin{matrix} n+m+5/4,\, n-m+5/4 \\ n-k+7/4 \end{matrix}; \tfrac{1}{2}\right\} \overset{k}{\underset{m}{=}} x^n,$$

$$\mathbf{R}(p) > 0 \text{ and } \mathbf{R}(n \pm m+5/4) > 0. \quad (1)$$

When $k = 0$, the function ${}_2F_1$ reduces, on account of the relation (Baily, (1935)

$$ {}_2F_1\left\{\begin{matrix} a,\, b \\ c \end{matrix}; \tfrac{1}{2}\right\} = \frac{\sqrt{\pi}.\Gamma\left(\frac{a+b+1}{2}\right)}{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)} \quad (A)$$

and, we have

$$\frac{\Gamma(n+m+5/4)\Gamma(n-m+5/4)}{2^{n+1}\Gamma(n+7/4)} . {}_2F_1\left\{\begin{matrix} n+m+5/4,\, n-m+5/4 \\ n+7/4 \end{matrix}; \tfrac{1}{2}\right\} p^{-n}$$

$$= \frac{2^{n-\frac{1}{2}}}{\sqrt{\pi}} \Gamma(\tfrac{1}{2}n+\tfrac{1}{2}m+5/8)\Gamma(\tfrac{1}{2}n-\tfrac{1}{2}m+5/8)p^{-n}$$

$$\overset{o}{\underset{m}{=}} x^n. \quad (1a)$$

If $k = \frac{1}{2}$, $m = \pm\frac{1}{2}$, we get the fundamental operational relation (McLachlan and Humbert),

$$p^{-n} \doteqdot \frac{x^n}{\Gamma(n+1)}, \quad \mathbf{R}(n+1) > 0 \quad (1b)$$

in symbolic calculus.*

Example 2. When $h(x) = x^n\,{}_1F_1[\alpha, \beta; -qx]$, we have, on expanding ${}_1F_1[\alpha; \beta; -qx]$ and integrating by the help of (D.2),†

$$\frac{\Gamma(\beta)}{\Gamma(\alpha)2(2p)^n} \sum_{r=0}^{\infty} \frac{(-1)^r\Gamma(\alpha+r)\Gamma(n+m+r+5/4)\Gamma(n-m+r+5/4)}{r!\,\Gamma(\beta+r)\Gamma(n-k+r+7/4)}\left(\frac{q}{2p}\right)^r$$

$$\times {}_2F_1\left\{\begin{matrix} n+m+r+5/4,\, n-m+r+5/4 \\ n-k+r+7/4 \end{matrix}; \tfrac{1}{2}\right\} \quad (2)$$

$$\overset{k}{\underset{m}{=}} x^n . {}_1F_1[\alpha; \beta; -qx], \quad \mathbf{R}(n \pm m+5/4) > 0, \quad \mathbf{R}(p) > 0 \text{ and } |p| > |q|.$$

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* We shall, henceforth, refer the book of Humbert and McLachlan as H. In this case see H, p. 14

† The convergence of the series (2) follows easily by considering the behaviour

$$F[a+r, b+r, c+r; x] < \frac{(c)_r}{(A)_r}(1-x)^{-r} F[a, b; c; x],$$

where $A = \min(a, b, c)$.

For $k=\frac{1}{2}$, $m=\pm\frac{1}{2}$, see H, p 54. Henceforth H., p. — , would mean, for $k=\frac{1}{2}$, $m=\pm\frac{1}{2}$, the result given in the book of Humbert.

If in (2) $\alpha = \beta$, we get, on using the result due to Buschnall and Chundy (1940)

$$F[a,\, b;\, c;\; x+y] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{m\,!\; n\,!\; (c)_{m+n}}\, x^m y^n \tag{B}$$

that

$$\frac{1}{2(2p)^n}\sum_{r=0}^{\infty} \frac{(-)^r\Gamma(n+m+r+5/4)\Gamma(n-m+r+5/4)}{r\,!\;\Gamma(n-k+r+7/4)}\left(\frac{q}{2p}\right)^r \cdot {}_2F_1\left\{\begin{matrix} n+m+r+5/4,\; n-m+r+5/4 \\ n-k+r+7/4 \end{matrix};\; \tfrac{1}{2}\right\}$$

$$= \frac{\Gamma(n+m+5/4)\Gamma(n-m+5/4)}{2(2p)^n\Gamma(n-k+7/4)} \cdot {}_2F_1\left\{\begin{matrix} n+m+5/4,\; n-m+5/4 \\ n-k+7/4 \end{matrix};\; \frac{1}{2}-\frac{q}{2p}\right\} \tag{3}$$

$$\overset{k}{\underset{m}{=}}\; x^n e^{-qx}, \quad \mathbf{R}(n \pm m+5/4) > 0, \quad \mathbf{R}(p) > 0 \quad \text{and} \quad |p| > |q|$$

Example 3. When $h(x) = x^\lambda e^{-\frac{1}{2}x^2}$, we have

$$\varphi(p) = p\int_0^\infty (2xp)^{-\frac{1}{4}} W_{k,m}(2px)x^\lambda e^{-\frac{1}{2}x^2}\,dx.$$

Writing the two infinite series for the Whittaker function given by (D.1) and integrating term by term, which is justified, we get, on using Whittaker's integral for Weber's parabolic cylinder function,

$$D_{-n}(z) = \frac{e^{-\frac{1}{4}z^2}}{\Gamma(n)}\int_0^\infty e^{-tz-\frac{1}{2}t^2}t^{n-1}\,dt, \quad \mathbf{R}(n) > 0 \tag{c}$$

that (H., p. 17)

$$\frac{pe^{\frac{1}{4}p^2}}{\Gamma(\frac{1}{2}-m-k)\Gamma(\frac{1}{2}+m-k)}\Bigg[\Gamma(-2m)\Gamma(2m+1)\sum_{r=0}^{\infty}\frac{\Gamma(m+\frac{1}{2}-k+r)\Gamma(\lambda+m+r+5/4)}{r\,!\;\Gamma(2m+1+r)}(2p)^{m+r+\frac{1}{4}}$$

$$\times D_{-(\lambda+m+r+5/4)}(p) + \Gamma(2m)\Gamma(-2m+1)\sum_{r=0}^{\infty}\frac{\Gamma(-m+\frac{1}{2}-k+r)\Gamma(\lambda-m+r+5/4)}{r\,!\;\Gamma(-2m+1+r)}$$

$$\times (2p)^{-m+r+\frac{1}{4}}D_{-(\lambda-m+r+5/4)}(p)\Bigg]$$

$$\overset{k}{\underset{m}{=}}\; x^\lambda e^{-\frac{1}{2}x^2}$$

provided $(2m)$ is not an integer and $\mathbf{R}(\lambda \pm m+5/4) > 0$.

Example 4. When $h(x) = x^{-n}e^{-a/x}$, we have, on using the integral for Bessel function (Watson, 1922) of the second kind for imaginary argument, (H., p. 16; B.F., p. 183),

$$K_\nu(z) = \tfrac{1}{2}(\tfrac{1}{2}z)^\nu\int_0^\infty e^{-x-z^2/4x}\frac{dx}{x^{\nu+1}}, \quad \mathbf{R}(z^2) > 0 \tag{D}$$

$$\frac{2p^n}{\Gamma(\frac{1}{2}-m-k)\Gamma(\frac{1}{2}+m-k)}\Bigg[\Gamma(-2m)\Gamma(2m+1)\sum_{r=0}^{\infty}\frac{2^{m+r+\frac{1}{4}}\Gamma(m+\frac{1}{2}-k+r)}{r\,!\;\Gamma(2m+1+r)}(ap)^{\frac{1}{2}(-n+m+r+5/4)}$$

$$\times K_{n-m-r-5/4}\{2\sqrt{(ap)}\} + \Gamma(2m)\Gamma(-2m+1)\sum_{r=0}^{\infty}\frac{2^{-m+r+\frac{1}{4}}\Gamma(-m+\frac{1}{2}-k+r)}{r\,!\;\Gamma(-2m+1+r)}$$

$$\times (ap)^{\frac{1}{2}(-n-m+r+5/4)}K_{n+m-r-5/4}\{2\sqrt{(ap)}\}\Bigg]$$

$$\overset{k}{\underset{m}{=}}\; x^{-n}e^{-a/x}. \tag{5}$$

provided $(2m)$ is not an integer, $\mathbf{R}(4ap) \gtrsim 0$.

Example 5. When $h(x) = x^{\nu/2}\exp(-\sqrt{x})$, we obtain, as in (Ex. 3),

$$\frac{\Gamma(-2m)\Gamma(2m+1)}{\Gamma(\frac{1}{2}-m-k)\Gamma(\frac{1}{2}+m-k)}\sum_{r=0}^{\infty}\frac{\Gamma(r+m+\frac{1}{2}-k)\Gamma(2m+\nu+2r+5/2)}{r!\,\Gamma(r+2m+1)(2p)^{\nu/2}}e^{1/8p}D_{-(2m+\nu+2r+5/2)}\{1/\sqrt{(2p)}\}$$

$$+\frac{\Gamma(2m)\Gamma(-2m+1)}{\Gamma(\frac{1}{2}-m-k)\Gamma(\frac{1}{2}+m-k)}\sum_{r=0}^{\infty}\frac{\Gamma(r-m+\frac{1}{2}-k)\Gamma(-2m+\nu+2r+5/2)}{r!\,\Gamma(r-2m+1)(2p)^{\nu/2}}$$

$$\times e^{1/8p}D_{-(-2m+\nu+2r+5/2)}\{1/\sqrt{(2p)}\}$$

$$\overset{k}{\underset{m}{=}}\; x^{\nu/2}\exp(-\sqrt{x}) \tag{6}$$

(H., p. 17) provided $(2m)$ is not an integer and $\mathbf{R}(\pm 2m+\nu+5/2)>0$.

Example 6. When $h(x) = x^{n/2}J_n(2\sqrt{x})$, we obtain from (D. 2), (H., p. 28)

$$\frac{1}{2}\sum_{r=0}^{\infty}\frac{(-)^r\Gamma(n+m+r+5/4)\Gamma(n-m+r+5/4)}{r!\,\Gamma(n+r+1)\Gamma(n-k+r+7/4)(2p)^{n+r}}\cdot{}_2F_1\left\{\begin{matrix}n+m+r+5/4,\; n-m+r+5/4\\ n-k+r+7/4\end{matrix};\frac{1}{2}\right\}$$

$$\overset{k}{\underset{m}{=}}\; x^{n/2}J_n(2\sqrt{x}),\quad \mathbf{R}(n\pm m+5/4)>0$$

Example 7. When $h(x) = x^{\mu}e^{-ax}J_n(bx)$, we obtain, (H., p. 28) from (D. 2)

$$\frac{b^n}{2^{n+1}(2p)^{\mu+n}}\sum_{r=0}^{\infty}\frac{(-)^r\Gamma(n+\mu+m+2r+5/4)\Gamma(n+\mu-m+2r+5/4)}{r!\,\Gamma(n+1+r)\Gamma(n+\mu-k+2r+7/4)}\left(\frac{b^2}{16p^2}\right)^r$$

$$\times{}_2F_1\left\{\begin{matrix}n+\mu+m+2r+5/4,\; n+\mu-m+2r+5/4\\ n+\mu-k+2r+5/4\end{matrix};\frac{1}{2}-\frac{a}{2p}\right\}$$

$$\overset{k}{\underset{m}{=}}\; x^{\mu}e^{-ax}J_n(bx) \tag{9}$$

provided $\mathbf{R}(n+\mu\pm m+5/4)>0$, $\mathbf{R}(a+p)>0$ and $|a+p|>|b|$.

Example 8. When $h(x) = J_{\mu}(x)J_{\nu}(x)$, we obtain, from (D. 2) and

$$J_{\mu}(x)J_{\nu}(x) = \frac{(x/2)^{\mu+\nu}}{\Gamma(\mu+1)\Gamma(\nu+1)}\cdot{}_2F_3\left\{\begin{matrix}\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2},\; \frac{1}{2}\mu+\frac{1}{2}\nu+1\\ \mu+1,\; \nu+1,\; \mu+\nu+1\end{matrix};-x^2\right\} \tag{E}$$

that

$$\frac{(4p)^{-\mu-\nu}}{2\Gamma(\mu+1)\Gamma(\nu+1)}\sum_{r=0}^{\infty}\frac{(-)^r(\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2},\, r)(\frac{1}{2}\mu+\frac{1}{2}\nu+1,\, r)}{r!\,(\mu+1,\, r)(\nu+1,\, r)(\mu+\nu+1,\, r)}\cdot\frac{\Gamma(\mu+\nu+m+2r+5/4)\Gamma(\mu+\nu-m+2r+5/4)}{\Gamma(\mu+\nu-k+2r+7/4)(2p)^{2r}}$$

$$\times{}_2F_1\left\{\begin{matrix}\mu+\nu+m+2r+5/4,\; \mu+\nu-m+2r+5/4\\ \mu+\nu-k+2r+7/4\end{matrix};\frac{1}{2}\right\}$$

$$\overset{k}{\underset{m}{=}}\; J_{\mu}(x)J_{\nu}(x),\; \mathbf{R}(\mu+\nu\pm m+5/4)>0,\; \mathbf{R}(p)>0 \text{ and } |p|>2. \tag{10}$$

Example 9. When

$$h(x) = x^{h}J_{m_1,n}(x),$$

where

$$J_{m_1,n}(x) = \frac{x^{m_1+n}}{3^{m_1+n}\Gamma(m_1+1)\Gamma(n+1)}\cdot{}_0F_2[m_1+1,\, n+1;\; -x^3/27],$$

we obtain from (D. 2), provided that $\mathbf{R}(p)=0$, $|p|>1$, and $\mathbf{R}(m_1+n+h\pm m+5/4)>0$;

$$\frac{(2p)^{-(m_1+n+h)}}{2.3^{m_1+n}}\sum_{r=0}^{\infty}\frac{(-)^r\Gamma(m_1+m+n+h+3r+5/4)\Gamma(m_1-m+n+h+3r+5/4)(6p)^{-3r}}{r!\,\Gamma(m_1+r+1)\Gamma(n+r+1)\Gamma(m_1-k+n+h+3r+7/4)}$$

$$\times\,{}_2F_1\left\{\begin{matrix}m_1+m+n+h+3r+5/4,\;\; m_1-m+n+h+3r+5/4\\ m_1-k+n+h+3r+7/4\end{matrix};\,\tfrac{1}{2}\right\}$$

$$\overset{k}{\underset{m}{=}}\; x^h J_{m_1,n}(x),^{*}\;\text{(H., p. 48).} \tag{11}$$

Example 10. When $h(x) = K_\nu(x/2)$, we obtain from (D. 1) and (D. 2) (H., p. 39)

$$\frac{\sqrt{\pi}.p}{\Gamma(\frac{1}{2}+m-k)\Gamma(\frac{1}{2}-m-k)}\Bigg[\Gamma(-2m)\Gamma(2m+1)\sum_{r=0}^{\infty}\frac{\Gamma(m+\frac{1}{2}-k+r)\Gamma(m+\nu+r+5/4)\Gamma(m-\nu+r+5/4)}{r!\,\Gamma(2m+1+r)\Gamma(m+r+7/4)}$$

$$\times\,(2p)^{m+\nu+\frac{1}{2}}.{}_2F_1\left\{\begin{matrix}m+\nu+r+5/4,\;\; m-\nu+r+5/4\\ m+r+7/4\end{matrix};\,\tfrac{1}{2}-p\right\}$$

$$+\,\Gamma(2m)\Gamma(-2m+1)\sum_{r=0}^{\infty}\frac{\Gamma(-m+\frac{1}{2}-k+r)\Gamma(\nu-m+r+5/4)\Gamma(-\nu-m+r+5/4)}{r!\,\Gamma(-2m+1+r)\Gamma(-m+r+7/4)}$$

$$\times\,(2p)^{-m+\nu+\frac{1}{2}}.{}_2F_1\left\{\begin{matrix}-m+\nu+r+5/4,\;\; -m-\nu+r+5/4\\ -m+r+7/4\end{matrix};\,\tfrac{1}{2}-p\right\}$$

$$\overset{k}{\underset{m}{=}}\; K_\nu(x/2). \tag{12}$$

provided $(2m)$ is not an integer $\mathbf{R}(\pm\nu\pm m+5/4) > 0$ and $-\frac{1}{2} < \mathbf{R}(p) < \frac{1}{2}$.

Example 11. Since

$$H_\nu(2x) = x^{\nu+1}\sum_{r=0}^{\infty}\frac{(-)^r x^{2r}}{\Gamma(r+3/2)\Gamma(\nu+r+3/2)}$$

(The function of Struve),

$$L_\nu(2x) = x^{\nu+1}\sum_{r=0}^{\infty}\frac{x^{2r}}{\Gamma(r+3/2)\Gamma(\nu+r+3/2)}$$

(The function of Struve for imaginary argument),

$$I_\nu(2x) = x^{\nu}\sum_{r=0}^{\infty}\frac{x^{2r}}{\Gamma(r+1)\Gamma(\nu+r+1)}$$

(Bessel function for imaginary argument).

we easily obtain the following Whittaker transforms:

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[*] The convergence of the series of the first line of (11) can be established by using the definite integral for ${}_2F_1$, *viz*,

$$F[a,\,b\,;\,c\,;\,z] = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt$$

$$(a)^{*}\quad \tfrac{1}{2}\sum_{r=0}^{\infty}\frac{(-)^r\Gamma(\nu+m+2r+9/4)\Gamma(\nu-m+2r+9/4)}{\Gamma(r+3/2)\Gamma(\nu+r+3/2)\,\Gamma(\nu-k+2r+11/4)(2p)^{\nu+2r+1}}$$

$$\times\,{}_2F_1\left\{\begin{matrix}\nu+m+2r+9/4,\ \ \nu-m+2r+9/4\\ \nu-k+2r+11/4\end{matrix};\ \tfrac{1}{2}\right\}$$

$$\underset{m}{\overset{k}{=}}\ H_\nu(2x),\ \ \mathbf{R}(\nu\pm m+9/4)>0,\ \ \mathbf{R}(p)>2,\ \text{and}\ |p|>2. \tag{13}$$

$$(b)\quad \tfrac{1}{2}\sum_{r=0}^{\infty}\frac{\Gamma(\nu+m+2r+9/4)\Gamma(\nu-m+2r+9/4)}{\Gamma(r+3/2)\Gamma(\nu+r+3/2)\Gamma(\nu-k+2r+11/4)}(2p)^{-(\nu+2r+1)}$$

$$\times\,{}_2F_1\left\{\begin{matrix}\nu+m+2r+9/4,\ \ \nu-m+2r+9/4\\ \nu-k+2r+11/4\end{matrix};\ \tfrac{1}{2}\right\}$$

$$\underset{m}{\overset{k}{=}}\ L_\nu(2x),\ \ \mathbf{R}(\nu\pm m+9/4)>0,\ \ \mathbf{R}(p)>0\ \text{and}\ \ |p|>2 \tag{14}$$

(c) (H., p. 35)

$$\tfrac{1}{2}\sum_{r=0}^{\infty}\frac{\Gamma(\nu+m+2r+5/4)\Gamma(\nu-m+2r+5/4)}{r!\ \Gamma(\nu+r+1)\Gamma(\nu-k+2r+7/4)(2p)^{\nu+2r}}\cdot{}_2F_1\left\{\begin{matrix}\nu+m+2r+5/4,\ \ \nu-m+2r+5/4\\ \nu-k+2r+7/4\end{matrix};\ \tfrac{1}{2}\right\}$$

$$\underset{m}{\overset{k}{=}}\ I_\nu(2x),\quad \mathbf{R}(\nu\pm m+5/4)>0\ \ \mathbf{R}(p)>0\ \text{and}\ |p|>2 \tag{15}$$

Example 12. When $h(x)=x^{\lambda}W_{\mu,\nu}(x)$, we obtain from (D.1) and (D. 2)

$$p\frac{\Gamma(-2m)\Gamma(2m+1)}{\Gamma(\frac{1}{2}-m-k)\Gamma(\frac{1}{2}+m-k)}\sum_{r=0}^{\infty}\frac{\Gamma(\frac{1}{2}+m-k+r)\Gamma(\lambda+m+\nu+r+7/4)\Gamma(\lambda+m-\nu+r+7/4)}{r!\ \Gamma(2m+1+r)\Gamma(\lambda+m-\mu+r+9/4)}$$

$$\times\,(2p)^{m+r+\frac{1}{4}}\cdot{}_2F_1\left\{\begin{matrix}\lambda+m+\nu+r+7/4,\ \ \lambda+m-\nu+r+7/4\\ \lambda+m-\mu+r+9/4\end{matrix};\ \tfrac{1}{2}-p\right\}$$

$$+\,p\frac{\Gamma(2m)\Gamma(-2m+1)}{\Gamma(\frac{1}{2}+m-k)\Gamma(\frac{1}{2}-m-k)}\sum_{r=0}^{\infty}\frac{\Gamma(\frac{1}{2}-m-k+r)\Gamma(\lambda-m+\nu+r+7/4)\Gamma(\lambda-m-\nu+r+7/4)}{r!\ \Gamma(-2m+r+1)\Gamma(\lambda-m-\mu+r+9/4)}$$

$$\times\,(2p)^{-m+r+\frac{1}{4}}\cdot{}_2F_1\left\{\begin{matrix}\lambda-m+\nu+r+7/4,\ \ \lambda-m-\nu+r+7/4\\ \lambda-m-\mu+r+9/4\end{matrix};\ \tfrac{1}{2}-p\right\}$$

$$\underset{m}{\overset{k}{=}}\ x^{\lambda}W_{\mu,\nu}(x) \tag{16}$$

provided $(2m)$ is not an integer and $\mathbf{R}(\lambda\pm m\pm\nu+7/4)>0$ and $-\frac{1}{2}<\mathbf{R}(p)<\frac{1}{2}$.

Example 13. Since

$$D_n\{\sqrt{(2x)}\}=2^{\frac{1}{2}n}x^{-\frac{1}{4}}W_{\frac{1}{2}n+\frac{1}{4},\pm\frac{1}{4}}(x).$$

We have, from (16), with $\lambda=-\frac{1}{4}$,

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[*] The convergence of the series follows easily by considering the behaviour

$${}_2F_1\left\{\begin{matrix}a+2r,\ 2b+2r\\ c+2r\end{matrix};\ x\right\}<\frac{(c)_{2r}}{(b)_r\,(b)_r}(1-x)^{-2r}\cdot{}_2F_1\left\{\begin{matrix}a,\ 2b\\ c\end{matrix};\ x\right\},$$

where $(a)_r\equiv\Gamma(a+r)/\Gamma(a)$.

$$\frac{\Gamma(-2m)\Gamma(2m+1)}{\Gamma(\tfrac{1}{2}+m-k)\Gamma(\tfrac{1}{2}-m-k)}p\sum_{r=0}^{\infty}\frac{\Gamma(r+\tfrac{1}{2}+m-k)\Gamma(m+r+7/4)\Gamma(m+r+5/4)}{r!\,\Gamma(2m+1+r)\Gamma(m-\tfrac{1}{2}n+r+7/4)}(2p)^{m+r+\frac{1}{2}}$$

$$\times\,{}_2F_1\left\{\begin{matrix}m+r+7/4,\ m+r+5/4\\ m-\tfrac{1}{2}n+r+7/4\end{matrix};\ \tfrac{1}{2}-p\right\}$$

$$+\frac{\Gamma(2m)\Gamma(-2m+1)}{\Gamma(\tfrac{1}{2}+m-k)\Gamma(\tfrac{1}{2}-m-k)}p\sum_{r=0}^{\infty}\frac{\Gamma(\tfrac{1}{2}-m-k+r)\Gamma(-m+r+7/4)\Gamma(-m+r+5/4)(2p)^{-m+r+\frac{1}{2}}}{r!\,\Gamma(-2m+1+r)\Gamma(-m-\tfrac{1}{2}n+r+7/4)}$$

$$\times\,{}_2F_1\left\{\begin{matrix}-m+r+7/4,\ -m+r+5/4\\ -m-\tfrac{1}{2}n+r+7/4\end{matrix};\ \tfrac{1}{2}-p\right\}$$

$$\overset{k}{\underset{m}{=}}\ 2^{-\frac{1}{2}n}D_n\{\sqrt{(2x)}\},\quad -\tfrac{1}{2}<\mathbf{R}(p)<\tfrac{1}{2}\ \text{ and }\ \mathbf{R}(\pm m+5/4)>0 \tag{17}$$

Example 14. When

$$h(x) = x^{3\lambda/2-3/2}e^{\frac{1}{2}x}W_{-\nu,\lambda}(x),$$

we obtain, from (D. 1) and (D. 2),

$$\frac{1}{\Gamma(\tfrac{1}{2}+\nu+\lambda)\Gamma(\tfrac{1}{2}+\nu-\lambda)}p\Bigg[\frac{\Gamma(-2\lambda)\Gamma(2\lambda+1)}{2^{\lambda+\frac{1}{2}}}(2p)^{-5\lambda/2}$$

$$\times\sum_{r=0}^{\infty}\frac{\Gamma(r+\nu+\tfrac{1}{2}+\lambda)\Gamma(5\lambda/2+m+r+\tfrac{1}{4})\Gamma(5\lambda/2-m+r+\tfrac{1}{4})}{r!\,\Gamma(r+2\lambda+1)\Gamma(5\lambda/2-k+r+3/4)(4p^r}$$

$$\times\,{}_2F_1\left\{\begin{matrix}5\lambda/2+m+r+\tfrac{1}{4},\ 5\lambda/2-m+r+\tfrac{1}{4}\\ 5\lambda/2-k+r+3/4\end{matrix},\ \tfrac{1}{2}\right\}$$

$$+\frac{\Gamma(2\lambda)\Gamma(-2\lambda+1)(2p)^{-\frac{1}{2}\lambda}}{2^{-\lambda+\frac{1}{2}}}\sum_{r=0}^{\infty}\frac{\Gamma(r-\lambda+\tfrac{1}{2}+\nu)\Gamma(\tfrac{1}{2}\lambda+m+r+\tfrac{1}{4})\Gamma(\tfrac{1}{2}\lambda-m+r+\tfrac{1}{4})}{r!\,\Gamma(r-2\lambda+1)\Gamma(\tfrac{1}{2}\lambda-k+r+3/4)(4p)^r}$$

$$\times\,{}_2F_1\left\{\begin{matrix}\tfrac{1}{2}\lambda+m+r+\tfrac{1}{4},\ \tfrac{1}{2}\lambda-m+r+\tfrac{1}{4}\\ \tfrac{1}{2}\lambda-k+r+3/4\end{matrix};\ \tfrac{1}{2}\right\}$$

$$\overset{k}{\underset{m}{=}}\ x^{3\lambda/2-3/2}e^{\frac{1}{2}x}W_{-\nu,\lambda}(x)\Bigg],\quad \mathbf{R}(3\lambda/2\pm m\pm\lambda\pm\tfrac{1}{4})>0,\ \mathbf{R}(p)>0\ \text{ and }\ |p|>\tfrac{1}{2}. \tag{11}$$

Example 15. When*

$$h(x) = X_n^b\{2\sqrt{(2x)}\}$$

$$= \mathrm{ber}_n^2\{2\sqrt{(2x)}\}+\mathrm{bei}_n^2\{2\sqrt{(2x)}\}$$

where

$$X_n^b\{2\sqrt{(2x)}\} = \frac{2^n x^n}{n!\ n!}\cdot{}_0F_3[n+1,\ \tfrac{1}{2}n+1,\ \tfrac{1}{2}n+\tfrac{1}{2};\ x^2]$$

ber(x) and bei(x) being Kelvin's functions. We obtain, from (D 2), that

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* It is easy to see from the asymptotic expansions of ber(z) and bei(z) that $X_n^b\{2\sqrt{(2x)}\}=O\{x^{-\frac{1}{2}}\exp(4\sqrt{x})\}$ for large x.

$$\frac{1}{2p^n n!\, n!}\sum_{r=0}^{\infty}\frac{\Gamma(n+m+2r+5/4)\Gamma(n-m+2r+5/4)(4p^2)^{-r}}{r!\,(n+1,\, r)(\tfrac{1}{2}n+1,\, r)(\tfrac{1}{2}n+\tfrac{1}{2},\, r)\Gamma(n-k+2r+7/4)}$$

$$\times\, {}_2F_1\left\{\begin{matrix} n+m+2r+5/4,\ n-m+2r+5/4 \\ n-k+2r+7/4 \end{matrix};\ \tfrac{1}{2}\right\}$$

$$\overset{k}{\underset{m}{=}} X_n^b\{2\surd(2x)\},\quad \mathbf{R}(n\pm m+5/4)>0 \text{ and } \mathbf{R}(p)>0. \tag{19}$$

Example 16. When

$$h(x) = x^{\frac{1}{2}(\lambda-1)}(1+x)^{-n-\frac{1}{2}}.$$

We obtain from (D. 1) and the integral (Whittaker and Watson, 1927, p. 340) .

$$\mathrm{W}_{k,m}(z) = \frac{e^{-\frac{1}{2}z}z^k}{\Gamma(\frac{1}{2}-k+m)}\int_0^{\infty} t^{-k-\frac{1}{2}+m}(1+t/z)^{k-\frac{1}{2}+m}e^{-t}dt,\quad \mathbf{R}(k-m-\tfrac{1}{2})\leq 0. \tag{F}$$

$$p\frac{\Gamma(-2m)\Gamma(2m+1)}{\Gamma(\frac{1}{2}-m-k)\Gamma(\frac{1}{2}+m-k)}\sum_{r=0}^{\infty}\frac{\Gamma(r+m+\frac{1}{2}-k)\Gamma(\frac{1}{2}\lambda+m+r+3/4)(2p)^{m+r+\frac{1}{2}}}{r!\,\Gamma(r+2m+1)}$$

$$\times\, e^{\frac{1}{2}p}p^{-\frac{1}{2}(\frac{1}{2}\lambda+m-n+r+5/4)}W_{\left\{\substack{-\frac{1}{2}(\frac{1}{2}\lambda+m+n+r+\frac{1}{4}) \\ \frac{1}{2}(\frac{1}{2}\lambda+m-n+r+\frac{1}{4})}\right\}}(p)$$

$$+\,p\frac{\Gamma(2m)\Gamma(-2m+1)}{\Gamma(\frac{1}{2}-m-k)\Gamma(\frac{1}{2}+m-k)}\sum_{r=0}^{\infty}\frac{\Gamma(r-m+\frac{1}{2}-k)\Gamma(\frac{1}{2}\lambda-m+r+3/4)(2p)^{-m+r+\frac{1}{2}}}{r!\,\Gamma(r-2m+1)},$$

$$\times\, e^{\frac{1}{2}p}p^{-\frac{1}{2}(\frac{1}{2}\lambda-m-n+r+5/4)}W_{\left\{\substack{-\frac{1}{2}(\frac{1}{2}\lambda-m+n+r+\frac{1}{4}) \\ \frac{1}{2}(\frac{1}{2}\lambda-m-n+r+\frac{1}{4})}\right\}}(p)$$

$$\overset{k}{\underset{m}{=}} x^{\frac{1}{2}(\lambda-1)}(1+x)^{-n-\frac{1}{2}},\quad \mathbf{R}(p)>0 \text{ and } \mathbf{R}(\tfrac{1}{2}\lambda\pm m+3/4)>0. \tag{20}$$

Example 17. When $h(x) = e^{-1/x}x^{\lambda}(1+x)^{\mu}$, were μ is a positive integer, we obtain from (D. 1) and integrating term by term by the help of (1.2.D),

$$2p\frac{\Gamma(-2m)\Gamma(2m+1)}{\Gamma(\frac{1}{2}-m-k)\Gamma(\frac{1}{2}+m-k)}\sum_{r=0}^{\infty}\frac{\Gamma(m+\frac{1}{2}-k+r)}{r!\,\Gamma(r+2m+1)}(2p)^{m+r+\frac{1}{2}}$$

$$\times\sum_{t=0}^{\mu}\frac{1}{t!}\binom{\mu}{t}p^{-\frac{1}{2}(\lambda+m+r+t+5/4)}K_{-(\lambda+m+r+t+5/4)}(2p^{\frac{1}{2}})$$

$$+\,2p\frac{\Gamma(2m)\Gamma(-2m+1)}{\Gamma(\frac{1}{2}-m-k)\Gamma(\frac{1}{2}+m-k)}\sum_{r=0}^{\infty}\frac{\Gamma(-m+\frac{1}{2}-k+r)}{r!\,\Gamma(r-2m+1)}(2p)^{-m+r+\frac{1}{2}}$$

$$\times\sum_{t=0}^{\mu}\frac{1}{t!}\binom{\mu}{t}p^{-\frac{1}{2}(\lambda-m+r+t+5/4)}K_{-(\lambda-m+r+t+5/4)}(2p^{\frac{1}{2}})$$

$$\overset{k}{\underset{m}{=}} e^{-1/x}x^{\lambda}(1+x)^{\mu},\quad \mathbf{R}(p)>0. \tag{21}$$

II

2.1 In this section I have generalised some of the rules of the Operational Calculus for our new transform,

2.2. If $\varphi(p) \overset{k}{\underset{m}{=}} h(x)$, then

$$\varphi(p/s) \overset{k}{\underset{m}{=}} h(xs) \tag{R.1}$$

we have

$$\varphi(p) = p\int_0^\infty (2xp)^{-\frac{1}{2}} W_{k,m}(2px)h(x)dx.$$

If we first replace p by p/s and then x by xs, we obtain (R.1)

2.3. If $\varphi(p) \overset{k}{\underset{m}{=}} h(x)$, then

$$\left(-p\frac{d}{dp}\right)^n\{\varphi(p)\} \overset{k}{\underset{m}{=}} \left(x\frac{d}{dx}\right)^n\{h(x)\}.$$

On differentiating both sides of (R 1) with respect to s and then putting $s = 1$, we obtain

$$-p\frac{d}{dp}\{\varphi(p)\} \overset{k}{\underset{m}{=}} x\frac{d}{dx}\{h(x)\}$$

A repeated differentiation with respect to s thus yields

$$\left(-p\frac{d}{dp}\right)^n\{\varphi(p)\} \overset{k}{\underset{m}{=}} \left(x\frac{d}{dx}\right)^n\{h(x)\} \quad n > 0.$$

It being understood that both sides exist and are continuous.

2.4. If $\varphi(p) \overset{k}{\underset{m}{=}} h(x)$, then

$$\int_p^\infty \varphi(p)\frac{dp}{p} \overset{k}{\underset{m}{=}} \int_0^x h(x)\frac{dx}{x}, \tag{R.3}$$

Divide both sides of (R.1) by s and integrate between *zero* and *unity* with respect to s, we obtain

$$\int_p^\infty \varphi(p)\frac{dp}{p} \overset{k}{\underset{m}{=}} \int_0^x h(x)\frac{dx}{x}.$$

2.5. If $\varphi(p) \overset{k}{\underset{m}{=}} h(x)$ then

$$\int_0^\infty \varphi(p)\frac{dp}{p} \overset{k}{\underset{m}{=}} \int_0^\infty h(x)\frac{dx}{x}, \tag{R.4}$$

true when the integrals converge.

If the integration in §(2.4) is taken between the limits *zero* and *infinity*, we get (R.4).

2.6. If $\varphi(p) \overset{k}{\underset{m}{=}} h(x)$, then

$$\int_0^p \varphi(p)\frac{dp}{p} \overset{k}{\underset{m}{=}} \int_0^\infty h(x)\frac{dx}{x} \tag{R.5}$$

If we subtract (R 3) from (R.4), we obtain (R.5)

2.7. If $\varphi_1(p) \overset{k}{\underset{m}{=}} h_1(x)$ and $\varphi_2(p) \overset{k}{\underset{m}{=}} h_2(x)$, then

$$\int_0^\infty \varphi_1(u)h_2(u)\frac{du}{u} = \int_0^\infty h_1(t)\varphi_2(t)\frac{dt}{t} \tag{R.6}$$

provided that the order of integration may be inverted.

Here

$$\varphi_1(u) = u\int_0^\infty (2tu)^{-\frac{1}{4}}W_{k,m}(2tu)h_1(t)dt, \tag{2}$$

and

$$\varphi_2(t) = t\int_0^\infty (2tu)^{-\frac{1}{4}}W_{k,m}(2tu)h_2(u)du, \tag{3}$$

and therefore

$$\int_0^\infty \varphi_1(u)h_2(u)\frac{du}{u} = \int_0^\infty h_2(u)du\int_0^\infty (2tu)^{-\frac{1}{4}}W_{k,m}(2tu)h_1(t)dt$$

$$= \int_0^\infty h_1(t)\left[\int_0^\infty (2tu)^{-\frac{1}{4}}W_{k,m}(2tu)h_2(u)du\right]dt$$

$$= \int_0^\infty h_1(t)\varphi_2(t)\frac{dt}{t}$$

provided the change in the order of integration is permitted.

Regarding this change we note that the condition for the absolute convergence of (2) is $\mathbf{R}(\mu \pm m + 5/4) > 0$, where $h_1(t) = O(t^\mu)$ for small t and an additional condition, if required, $h_1(t) = O(e^{-t})^\lambda$, $\mathbf{R}(\lambda) > 0$ for large t. Integral (3) will be absolutely convergent if $\mathbf{R}(\mu_1 \pm m + 5/4) > 0$, where $h_2(u) = O(u^{\mu_1})$ for small u and an additional condition, if required, $h_2(u) = O\{\exp(-u^{\lambda_1})$, $\mathbf{R}(\lambda_1) > 0$ for large u.

We have imposed, however, hard conditions on $h_1(x)$ and $h_2(x)$ for the justification in the change of the order of integration. These conditions, however, may be relaxed.

2.8. If $\varphi(p) \overset{l}{\underset{m}{=}} h(x)$, then

$$\int_0^\infty \varphi(x)\frac{dx}{x} = \frac{\Gamma(5/4-m)\Gamma(5/4+m)}{2\Gamma(7/4-k)}\cdot {}_2F_1\left[\begin{matrix}5/4+m,\ 5/4-m\\ 7/4-k\end{matrix};\ \tfrac{1}{2}\right]\int_0^\infty h(x)\frac{dx}{x} \tag{R.7}$$

provided $\mathbf{R}(\mu \pm m + 5/4) > 0$, where $h(x) = O(x^\mu)$ for small x. Here

$$\varphi(x) = x\int_0^\infty (2xs)^{-\frac{1}{4}}W_{k,m}(2xs)h(s)ds. \tag{1}$$

Integrating both sides with respect to x from *zero* to *infinity* and on changing the order of integration and integrating by the help of (D.2), we get the above result.

Now as regards the justification of the change in the order of integration, (1) is absolutely convergent if $\mathbf{R}(\mu \pm m + 5/4) > 0$ were $h(s) = O(s^\mu)$ for small s and $s^{\frac{3}{4}}W_{k,m}(2xs)h(s)$ tends to zero uniformly as $s \to \infty$.

Also this result can be considered *a particular case* of (R 6), when we take $h_2 = 1$ and

$$\varphi_2 = \frac{\Gamma(5/4+m)\Gamma(5/4-m)}{2\Gamma(7/4-k)}\cdot {}_2F_1\left[\begin{matrix}5/4+m,\ 5/4-m\\ 7/4-k\end{matrix};\ \tfrac{1}{2}\right]$$

2.9. As an instance of (R. 3), we take (1.2.9) with $b = 1$, $n = \frac{1}{2}$, $\mu = \frac{1}{2}$ and $a = 0$,

$$\frac{\sqrt{\pi}}{8p}\sum_{r=0}^{\infty}\frac{(-)^r\Gamma(m+2r+9/4)\Gamma(-m+2r+9/4)}{r!\,\Gamma(r+3/2)\Gamma(-k+2r+11/4)(4p)^{2r}}\,{}_2F_1\left\{\begin{matrix}m+2r+9/4, & -m+2r+9/4\\ & -k+2r+11/4\end{matrix};\ \tfrac{1}{2}\right\}$$

$$\overset{k}{\underset{m}{=}}\sin x,\quad \mathbf{R}(\pm m+9/4)>0,\ \mathbf{R}(p)>0 \text{ and } |p|>1.$$

Applying (R. 3), we find on the one hand

$$\int_p^{\infty}\varphi(p)\frac{dp}{p} = \frac{\sqrt{\pi}}{8}\sum_{r=0}^{\infty}\frac{(-)^r\Gamma(m+2r+9/4)\Gamma(-m+2r+9/4)}{r!\,\Gamma(r+3/2)\Gamma(-k+2r+11/4)(2r+1)4^{2r}p^{2r+1}}$$

$$\times\ {}_2F_1\left\{\begin{matrix}m+2r+9/4, & -m+2r+9/4\\ & -k+2r+11/4\end{matrix};\ \tfrac{1}{2}\right\};$$

term by term integration is justified due to the uniform convergence of the infinite series for $\mathbf{R}(\pm m+5/4)>0$ and $|p|>1$. On the other hand, we have

$$\int_0^x h(x)\frac{dx}{x} = \int_0^x \sin x\frac{dx}{x} = \mathbf{Si}(x)$$

Thus, we find as the Whittaker transform for the "Integral sine."

$$\frac{\sqrt{\pi}}{8}\sum_{r=0}^{\infty}\frac{(-)^r\Gamma(m+2r+9/4)\Gamma(-m+2r+9/4)}{r!\,\Gamma(r+3/2)\Gamma(-k+2r+11/4)(2r+1)4^{2r}p^{2r+1}}\,{}_2F_1\left\{\begin{matrix}m+2r+9/4, & -m+2r+9/4\\ & -k+2r+11/4\end{matrix};\ \tfrac{1}{2}\right\}$$

$$\overset{k}{\underset{m}{=}}\mathbf{Si}(x),\quad \mathbf{R}(\pm m+5/4)>0,\ |p|>1 \text{ and } \mathbf{R}(p)>0.$$

2.10. As an instance of (R. 6), we take

$$h_1(x) = \frac{\sqrt{\pi}}{2^{2l-m+\frac{1}{2}}}x^{2l-m+\frac{3}{2}}e^{-x^2/4}T_l^n(x^2/4)$$

$$\overset{0}{\underset{m}{=}}\frac{(-)^n\Gamma(l-m+1)}{\Gamma(n+1)}p^{n+\frac{1}{2}}e^{\frac{1}{2}p^2}W_{\left\{\begin{smallmatrix}-\frac{1}{2}(n+2l-m+1)\\ -\frac{1}{2}(m+n)\end{smallmatrix}\right.}(p^2)$$

$$= \varphi_1(p),\quad \mathbf{R}(l+n)>-1 \text{ and } \mathbf{R}(l-m+n+1)>0$$

which can easily be deduced from the following integral of Shastri (1944)

$$\int_0^{\infty}e^{-t}t^{m-l}T_m^n(t)K_{2l}\{2\sqrt{(pt)}\}dt = \frac{(-)^n\Gamma(m-2l+1)}{2\Gamma(n+1)}p^{\frac{1}{2}(n-1)}e^{\frac{1}{2}p}W_{\left\{\begin{smallmatrix}-\frac{1}{2}(n+2m-2l+1),\\ -\frac{1}{2}(2l+n)\end{smallmatrix}\right.}(p),$$

$$\mathbf{R}(m+n)>-1,\ \mathbf{R}(m+n-2l+1)<0,$$

and $$h_2(x) = \sqrt{\pi}.x^{\nu+\frac{1}{2}}e^{-\frac{1}{4}x^2}$$

$$\overset{0}{\underset{m}{=}}\frac{\Gamma(\frac{1}{2}\nu+\frac{1}{2}m+7/8)\Gamma(\frac{1}{2}\nu-\frac{1}{2}m+7/8)}{2^{-\nu/2+3/8}}p^{\frac{1}{4}}e^{\frac{1}{4}p^2}W_{-\frac{1}{4}\nu-1/8,\frac{1}{2}m}(p^2/2)$$

$$= \varphi_2(p),\quad \mathbf{R}(\nu\pm m+7/4)>0,$$

which can easily be obtained from the following integral of Tewari (1945)

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$$p\int_0^\infty (2xp)^{\frac14}K_m(xp)x^{\nu+\frac14}e^{-\frac14x^2}dx = \frac{\Gamma(\frac12\nu+\frac12m+7/8)\Gamma(\frac12\nu-\frac12m+7/8)}{\Gamma(\nu+1)}\int_0^\infty (y/p)^{\nu+\frac14}y^{\nu+\frac14}e^{-\frac14y^2}$$

$$\times\, {}_2F_1\left\{\begin{matrix}\frac12\nu+\frac12m+7/8,\ \frac12\nu-\frac12m+7/8\\ \nu+1\end{matrix};\ -\frac{y^2}{p^2}\right\}dy,\quad \mathbf{R}(\nu\pm m+7/4)>0;$$

and the integral due to Meijer (1934)

$$\zeta^{-k}e^{\frac12\zeta}W_{k,m}(\zeta) = \frac{1}{\Gamma(a)}\int_0^\infty t^{a-1}e^{-t}\cdot{}_2F_1\left\{\begin{matrix}\frac12-k+m,\ \frac12-k-m\\ a\end{matrix};\ -t/\zeta\right\}dt \tag{A}$$

Hence, applying (R. 6) and using (D. 2), we find that

$$\int_0^\infty x^{2l-m}W_{\frac12\nu-3/8,\frac12m}(x^2/2)T_l^n(x^2/4)dx = \frac{2^{2l-m-\frac12\nu-5/8}(-)^n\Gamma(l-m+1)\Gamma(\frac12\nu+n+\frac12m+7/8)}{\Gamma(n+1)\Gamma(\frac12\nu+\frac12m+7/8)\Gamma(\frac12\nu+n+l-\frac12m+15/8)}$$

$$\times\, {}_2F_1\left\{\begin{matrix}\frac12\nu+n+\frac12m+7/8,\ \frac12\nu-\frac12m+7/8\\ \frac12\nu+n+l-\frac12m+15/8\end{matrix};\ \frac12\right\}$$

provided that, $\mathbf{R}(\nu\pm m+7/4)>0$, $\mathbf{R}(l+n)>-1$ and $\mathbf{R}(l-m+n+1)>0$.

III

3.1. We now give a theorem which will be useful in evaluating integrals involving our new function $G(s)$ introduced below. It will be seen that even our examples cover some of the known integrals deduced from the theory of Laplace's integral.

3.2. Theorem I. If

$$\varphi(p) \overset{k}{\underset{m_1}{\doteqdot}} h(x)$$

and

$$h(p)p^{1-m_1}(1+p)^{-l_1} \doteqdot g(x)$$

then, provided the integral is convergent,

$$(1/p)\varphi(p) = \int_0^\infty g(s)G(s)ds \tag{1}$$

where

$$G(s) = \frac{\Gamma(-2m)\Gamma(2m+1)}{\Gamma(\frac12-m-k)\Gamma(\frac12+m-k)}\sum_{r=0}^\infty \frac{\Gamma(r+m-k+\frac12)\Gamma(m_1+m+r+5/4)}{r!\,\Gamma(r+2m+1)}(2p)^{m+r+\frac14}$$

$$\times\, e^{\frac12(p+s)}(p+s)^{-\frac12(m_1+m+l_1+r+9/4)}W_{\frac12(l_1-m_1-m-r-\frac14),\ \frac12(m_1+m+l_1+r+5/4)}(p+s)$$

$$+\frac{\Gamma(2m)\Gamma(-2m+1)}{\Gamma(\frac12-m-k)\Gamma(\frac12+m-k)}\sum_{r=0}^\infty \frac{\Gamma(r-m-k+\frac12)\Gamma(m_1-m+r+5/4)}{r!\,\Gamma(r-2m+1)}e^{\frac12(p+s)}$$

$$\times\,(p+s)^{-\frac12(m_1-m+l_1+r+9/4)}(2p)^{-m+r+\frac14}W_{\frac12(l_1-m_1+m-r-\frac14),\ \frac12(m_1-m+l_1+r+5/4)}(p+s),$$

$(2m)$ and $(m_1\pm m+5/4)$ are not integers, $\mathbf{R}(m_1\pm m+5/4)>0$.

3.3. *Proof of Theorem I.* Here

$$\varphi(p) = p\int_0^\infty (2xp)^{-\frac14}W_{k,m}(2px)h(x)dx \tag{1}$$

and
$$h(x)x^{1-m_1}(1+x)^{-l_1} = x\int_0^\infty e^{-xs}g(s)ds \tag{2}$$

Substituting the value of $h(x)$ from (2) into (1) and on changing the order of integration, we get
$$\frac{1}{p}\varphi(p) = \int_0^\infty g(s)\Big[\int_0^\infty (2xp)^{-\frac{1}{4}}x^{m_1}(1+x)^{l_1}W_{k;m}(2px)e^{-xs}dx\Big]ds \tag{3}$$

Using (D. 1), this gives that
$$\frac{1}{p}\varphi(p) = \int_0^\infty g(s)\Big[\int_0^\infty (2xp)^{-\frac{1}{4}}x^{m_1}(1+x)^{l_1}e^{-xs}\Big\{\frac{\Gamma(-2m)\,(2xp)^{m+\frac{1}{2}}}{\Gamma(\frac{1}{2}-m-k)}.e^{-xp}{}_1F_1\left\{\begin{matrix} m+\frac{1}{2}-k \\ 2m+1\end{matrix};\ 2xp\right\}$$
$$+\frac{\Gamma(2m)\,(2xp)^{-m+\frac{1}{2}}}{\Gamma(\frac{1}{2}+m-k)}e^{-xp}.{}_1F_1\left\{\begin{matrix} -m+\frac{1}{2}-k \\ -2m+1\end{matrix};\ 2xp\right\}\Big\}dx\Big]ds.$$

Further expanding the hypergeometric function and integrating term by term, a process easily justifiable, by the help of (1.2.F.), we obtain (3.2.1) provided $(2m)$ and $(m_1 \pm m + 5/4)$ are not integers and $\mathbf{R}(m_1 \pm m + 5/4) > 0$. Now, in order to justify the change of order in (3) suppose that
$$\begin{aligned}\theta(s) &= g(s)\int_0^\infty (2xp)^{-\frac{1}{4}}x^{m_1}(1+x)^{l_1}W_{k,m}(2px)e^{-xs}dx \\ &= g(s)G(s)\end{aligned}$$

Now, I have shown below that
$$G(s) = O(1) \qquad \text{for small } s.$$

It is evident that $\theta(s)$ is uniformly convergent in $s \geq 0$, if $\mathbf{R}(\mu_1) > 0$, where $g(s) = O(s^{\mu_1})$ for small s.

Again, consider the integral
$$I = \int_0^\infty g(s)\int_A^\infty (2xp)^{-\frac{1}{4}}x^{m_1}(1+x)^{l_1}W_{k,m}(2px)e^{-xs}dx\,ds$$
where A is large.

Now, for large values of s, I does not exceed a constant multiple of
$$\int_0^\infty |g(s)|\Big[\int_0^\infty |(2xp)^{-\frac{1}{4}}x^{m_1}(1+x)^{l_1}W_{k,m}(2px)e^{-xs}|\,dx\Big]ds,$$
which tends to zero, if $\mathbf{R}(p) > 0$, on account of
$$W_{k,m}(2px) = O(x^k e^{-px}) \quad \text{for large } x,$$
the integral $\int_0^\infty e^{-xs}|g(s)|\,ds$ being supposed to be convergent.

Hence the change in the order of integration is justified under the conditions
$$\mathbf{R}(\mu_1) > 0, \quad \mathbf{R}(p) \geq 0,$$

$$\int_0^\infty |g(s)|\, e^{-xs} ds$$

being supposed to be convergent.

3.4. *Properties of $G(s)$.*

We have

$$G(s) = \int_0^\infty e^{-xs}(2px)^{-\frac{1}{4}}x^{m_1}(1+x)^{k_1}W_{k,m}(2px)dx$$

It follows from our definition of $G(s)$ that

$$G(s) = O(1) \qquad \text{for small } s.$$

If $(m_1 \pm m + 5/4)$ is not an integer, the function

$$x^{m_1-\frac{1}{4}}(1+x)^{k_1}W_{k,m}(2px)$$

satisfies all the conditions of Watson's lemma (B.F., p. 236) regarding the asymptotic expansion of function representable by Laplace's integral and is

$$O(x^{m_1 \pm m + \frac{1}{4}}) \qquad \text{for small } x.$$

We accordingly have from Watson's lemma

$$G(s) = O(s^{-m_1 \pm m - 5/4}) \quad \text{for large } s.$$

3.5. Corollary. If in (3.2 1), we put $k = \frac{1}{4}$, $m = \pm\frac{1}{4}$, Whittaker transform reduces to Laplace transform and the theorem reduces to Shastri's Theorem (1945)

3.6. *Examples.*

(i) If

$$g(s) = \frac{e^{-\frac{1}{2}s}\, s^{\mu+m_1-3/2}}{\Gamma(2m_1+2\mu-1)}M_{n+k_1,\, \mu+m_1-1}(s)$$

then (H., p. 50)

$$h(x)(1+x)^{-(k_1+m_1-\frac{1}{2})}x^{k_1-m_1+3/2} = \frac{x^{n-\mu+k_1-m_1+3/2}}{(1+x)^{n+\mu+k_1+m_1-\frac{1}{2}}}\ , \quad \mathbf{R}(\mu+m_1-\tfrac{1}{2}) > 0,$$

or

$$h(x) = x^{n-\mu}(1+x)^{-(n+\mu)}\frac{k}{m}\frac{\Gamma(-2m)\Gamma(2m+1)}{\Gamma(\frac{1}{2}-m-k)\Gamma(\frac{1}{2}+m-k)}(2p)^{m+\frac{1}{4}}$$

$$\times \sum_{r=0}^{\infty}\frac{\Gamma(r+m+\frac{1}{2}-k)\Gamma(n-\mu+m+r+5/4)}{r!\ \Gamma(r+2m+1)}2^r e^{\frac{1}{2}p}$$

$$\times\, p^{\mu-\frac{1}{2}m+\frac{1}{2}r-9/8}W_{\frac{1}{2}(-2n-m-r-\frac{1}{4})\ \frac{1}{2}(-2\mu+m+r+5/4)}(p)$$

$$+\frac{\Gamma(2m)\Gamma(-2m+1)}{\Gamma(\frac{1}{2}+m-k)\Gamma(\frac{1}{2}-m-k)}(2p)^{-m+\frac{1}{4}}\sum_{r=0}^{\infty}\frac{\Gamma(r-m+\frac{1}{2}-k)\Gamma(n-\mu-m+r+5/4)}{r!\ \Gamma(r-2m+1)}$$

$$\times\, 2^r e^{\frac{1}{2}p}p^{\mu+\frac{1}{2}m+\frac{1}{2}r-9/8}W_{\frac{1}{2}(-2n+m-r-\frac{1}{4})\ \frac{1}{2}(-2\mu-m+r+5/4)}$$

due to (1.2.20), provided $(2m)$ and $(n-\mu\pm m+5/4)$ are not integers, $\mathbf{R}(\pm m+\mu-n-5/4)\leq 0$.

Hence by (3.2.1) after replacing m_1 and l_1 by $m_1-k_1-\frac{1}{2}$ and $m_1+k_1-\frac{1}{2}$ respectively, we have

$$\frac{1}{\Gamma(2m_1+2\mu-1)}\int_0^\infty e^{-\frac{1}{2}s}s^{\mu+m_1-3/2}M_{n+k_1,\mu+m_1-1}(s)G(s)ds$$

$$=\frac{\Gamma(-2m)\Gamma(2m+1)}{\Gamma(\frac{1}{2}-m-k)\Gamma(\frac{1}{2}+m-k)}(2p)^{m+\frac{1}{4}}\sum_{r=0}^{\infty}\frac{\Gamma(r+m+\frac{1}{2}-k)\Gamma(n-\mu+m+r+5/4)}{r!\,\Gamma(r+2m+1)2^{-r}}$$

$$\times p^{\mu-\frac{1}{2}m+\frac{1}{2}r-9/8}e^{\frac{1}{2}p}W_{\frac{1}{2}(-2n-m-r-\frac{1}{4}),\frac{1}{2}(-2\mu+m+r+5/4)}(p)$$

$$+\frac{\Gamma(2m)\Gamma(-2m+1)}{\Gamma(\frac{1}{2}-m-k)\Gamma(\frac{1}{2}+m-k)}(2p)^{-m+\frac{1}{4}}\sum_{r=0}^{\infty}\frac{\Gamma(r-m+\frac{1}{2}-k)\Gamma(n-\mu-m+r+5/4)}{r!\,\Gamma(r-2m+1)2^{-r}}$$

$$\times p^{\mu+\frac{1}{2}m+\frac{1}{2}r-9/8}W_{\frac{1}{2}(-2n+m-r-\frac{1}{4}),\frac{1}{2}(-2\mu-m+r+5/4)}(p),$$

provided $(2m)$ and $(m_1-k_1\pm m+3/4)$ are not integers, $\mathbf{R}(\pm m+\mu-n-5/4)\leq 0$, $\mathbf{R}(\mu+m_1-\frac{1}{2})>0$ and $\mathbf{R}(m_1-k_1\pm m+3/4)>0$.

On putting $k=\frac{1}{4}$, $m=\pm\frac{1}{4}$, we get the result due to Shastri (1945)

(ii) If

$$g(s)=s^{\frac{1}{2}n}J_n(2\sqrt{s})$$

then (H., p. 28), l_1 being a positive integer

$$h(x)=e^{-1/x}x^{m_1-l_1-n-2}(1+x)^{l_1}\overset{k}{\underset{m}{=}}\frac{\Gamma(-2m)\Gamma(2m+1)}{\Gamma(\frac{1}{2}-m-k)\Gamma(\frac{1}{2}+m-k)}\sum_{r=0}^{\infty}\frac{\Gamma(r+m+\frac{1}{2}-k)(2p)^{m+r+\frac{1}{4}}}{r!\,\Gamma(r+2m+1)}$$

$$\times\sum_{t=0}^{l_1}\frac{1}{t!}\binom{l_1}{t}2p^{-\frac{1}{2}(m_1+m-l_1-n+r+t-\frac{1}{2})}K_{-(m_1+m-l_1-n+r+t-\frac{1}{2})}(2\sqrt{p})$$

$$+\frac{\Gamma(2m)\Gamma(-2m+1)}{\Gamma(\frac{1}{2}-m-k)\Gamma(\frac{1}{2}+m-k)}\sum_{r=0}^{\infty}\frac{\Gamma(r-m+\frac{1}{2}-k)(2p)^{-m+r+\frac{1}{4}}}{r!\,\Gamma(r-2m+1)}$$

$$\times\sum_{t=0}^{l_1}\frac{1}{t!}\binom{l_1}{t}2p^{-\frac{1}{2}(m_1-m-l_1-n+r+t-\frac{1}{2})}K_{-(m_1-m-l_1+r+t-\frac{1}{2})}(2\sqrt{p})$$

due to (1.2.21).

Hence by (3.2.1), after replacing m_1 by m_1-l_1-1, we get

$$\int_0^\infty s^{\frac{1}{2}n}J_n(2\sqrt{s})G(s)ds=2\frac{\Gamma(-2m)\Gamma(2m+1)}{\Gamma(\frac{1}{2}-m-k)\Gamma(\frac{1}{2}+m-k)}\sum_{r=0}^{\infty}\frac{\Gamma(r+m+\frac{1}{2}-k)(2p)^{m+r+\frac{1}{4}}}{r!\,\Gamma(r+2m+1)}$$

$$\times\sum_{t=0}^{l_1}\frac{1}{t!}\binom{l_1}{t}p^{-\frac{1}{2}(m_1+m-l_1-n+r+t-\frac{1}{2})}K_{-(m_1+m-l_1-n+r+t-\frac{1}{2})}(2\sqrt{p})$$

$$+\frac{2\Gamma(2m)\Gamma(-2m+1)}{\Gamma(\frac{1}{2}-m-k)\Gamma(\frac{1}{2}+m-k)}\sum_{r=0}^{\infty}\frac{\Gamma(r-m+\frac{1}{2}-k)}{r!\,\Gamma(r-2m+1)}(2p)^{-m+r+\frac{1}{4}}$$

$$\times\sum_{t=0}^{l_1}\frac{1}{t!}\binom{l_1}{t}p^{-\frac{1}{2}(m_1-m-l_1-n+r+t-\frac{1}{2})}K_{-(m_1-m-l_1-n+r+t-\frac{1}{2})}(2\sqrt{p})$$

provided $(2m)$ and $(m_1-l_1\pm m-\frac{3}{4})$ are not integers, $\mathbf{R}(m_1-l_1\pm m+\frac{1}{4})>0$ and $\mathbf{R}(p)>0$.

On putting $k = \frac{1}{4}$, $m = \frac{1}{4}$, we get the integral due to Shastri (1945).

(*iii*) If

$$g(s) = e^{-s}$$

then (H., p. 16)

$$h(x)x^{1-m_1}(1+x)^{-l_1} = \frac{x}{1+x}$$

or

$$h(x) = x^{m_1}(1+x)^{-\frac{1}{2}} \quad \text{if } l_1 = \tfrac{1}{2}.$$

Hence

$$\frac{1}{p}\varphi(p) = (-)^l l! \int_0^\infty e^{-px}(2px)^{\frac{1}{2}n+\frac{1}{4}} L_l^n(2px)x^{m_1}(1+x)^{-\frac{1}{2}}dx.$$

Since

$$L_l^n(2px = \Gamma(l+n+1)\sum_{r=0}^{\infty}\frac{(-2px)^r}{(l-r)!\,r!\,\Gamma(1+n+r)},$$

we have, on using the integral of Erdelyi (1938)

$$\int_0^\infty e^{-(1/s)x}x^{m-\frac{1}{2}}(1+x)^{-\frac{1}{2}}dx = \Gamma(m+\tfrac{1}{2})s^{\frac{1}{2}m+\frac{1}{4}}e^{1/2s}W_{-\frac{1}{2}m,\frac{1}{2}m}(1/s),$$

$$\frac{1}{p}\varphi(p) = (-)^l l!\,\Gamma(l+n+1)e^{\frac{1}{2}p}.\sum_{r=0}^{\infty}\frac{(-)^r\Gamma(m_1+\frac{1}{2}n+r+5/4)(2p)^{\frac{1}{2}n+r+\frac{1}{4}}}{(l-r)!\,r!\,\Gamma(1+n+r)p^{\frac{1}{2}(m_1+\frac{1}{2}n+r+7/4)}}W_{\left\{\begin{smallmatrix}-\frac{1}{2}(m_1+\frac{1}{2}n+r+\frac{1}{4})\\ \frac{1}{2}(m_1+\frac{1}{2}n+r+\frac{3}{4})\end{smallmatrix}\right.}(p)$$

Hence by using (3.2.1), with $k = \frac{1}{2}+\frac{1}{2}n+l$ and $m = \pm\frac{1}{2}n$, we get

$$\int_0^\infty e^{-s}G(s)ds = \sum_{r=0}^{\infty}\frac{(-)^r\Gamma(m_1+\frac{1}{2}n+r+5/4)(2p)^{\frac{1}{2}n+r+\frac{1}{4}}}{(l-r)!\,r!\,\Gamma(1+n+r)p^{\frac{1}{2}(m_1+\frac{1}{2}n+r+7/4)}}W_{\left\{\begin{smallmatrix}-\frac{1}{2}(m_1+\frac{1}{2}n+r+\frac{1}{4})\\ \frac{1}{2}(m_1+\frac{1}{2}n+r+\frac{3}{4})\end{smallmatrix}\right.}(p),$$

$$\mathbf{R}(m_1+\tfrac{1}{2}n+5/4) > 0.$$

(*iv*) If

$$g(s) = e^{-s}/\sqrt{s}$$

then (H., p. 16)

$$h(x)x^{1-m_1}(1+x)^{-l_1} = \frac{\sqrt{\pi}.x}{\sqrt{(1+x)}}$$

or

$$h(x) \equiv \sqrt{\pi}x^{m_1} \qquad \text{for } l_1 = \tfrac{1}{2}$$

$$\frac{k}{m}\sqrt{\pi}\,\frac{\Gamma(m_1+m+5/4)\Gamma(m_1-m+5/4)}{2^{m_1+1}\,\Gamma(m_1-k+7/4)p^{m_1}}\,{}_2F_1\left\{\begin{matrix}m_1+m+5/4, & m_1-m+5/4\\ & m_1-k+7/4\end{matrix};\ \tfrac{1}{2}\right\}$$

due to (1.2.1).

Hence by (3.2 1), we get

$$\int_0^\infty \frac{e^{-s}}{\sqrt{s}}G(s)ds = \sqrt{\pi}\,\frac{\Gamma(m_1+m+5/4)\,\Gamma(m_1-m+5/4)}{2^{m_1+1}\Gamma(m_1-k+7/4)p^{m_1}}\,{}_2F_1\left\{\begin{matrix}m_1+m+5/4, & m_1-m+5/4\\ & m_1-k+7/4\end{matrix};\ \tfrac{1}{2}\right\}$$

provided $(m_1 \pm m+5/4)$ is not an integer and $\mathbf{R}(m_1+m+5/4) > 0$.

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TORSION AND FLEXURE OF AN ISOTROPIC ELASTIC CYLINDER WHOSE CROSS-SECTION IS A SEMI-CARDIOD

By

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1. By representing conformally the area of the cross-section of a beam on the unit circle, Muschelisvili (1929) and Ghosh (1947*a*) have given explicit function-theoretic solutions of Saint-Venant's problems of torsion and flexure respectively. When the mapping function is of a complicated form, we meet with some difficulties in evaluating the integrals appearing in the solution. But when the cross-section is bounded partly by a straight line and can be mapped on the upper half of the unit circle by a function of simple form, the straight boundary corresponding to the bounding diameter of the semi-circle, the torsion-flexure problems can be solved by an extension of the Muschelisvili-Ghosh method (Ghosh, 1947*b* and 1948).

For such a boundary, Ghosh has reduced the problem to the determination of functions analytic within the upper half of the unit circle, real on the real axis and taking prescribed values on the semi-circumference. By Schwarz's principle of reflection, the functions have been extended to the lower half of the unit circle and have then been determined with the help of Schwarz's formula.

2. Let $z = \omega(\zeta)$ be the mapping function which gives a conformal representation of the area within the cardioid,

$$r = 2c(1-\cos\theta) \tag{2.1}$$

on the unit circle, so that the upper half of the region within the cardioid is mapped conformally on the upper half of the unit circle in the ζ-plane. Let α, β denote respectively the upper and lower halves of the circumference γ of the unit circle, and let a, b denote the cartesian coordinates of the centroid of half of the cardioid above the real axis.

Denoting the complex torsion function by $F_0(\zeta)$ instead of $G(\zeta)$ as has been done by Ghosh, $F_0(\zeta)$ and the two complex flexure functions $F_1(\zeta)$, $F_2(\zeta)$ are given by (Ghosh, 1947*b*, p. 108 and 1948, p. 78),

$$F_0(\zeta) = -\frac{1}{4\pi}\int_\alpha [\omega(t)-\bar{\omega}(1/t)]^2\frac{dt}{t-\zeta}+\frac{1}{4\pi}\int_\beta [\omega(t)-\bar{\omega}(1/t)]^2\frac{dt}{t-\zeta} \tag{2.2}$$

$$F_1(\zeta) = -\frac{\sigma b}{4\pi}\left[\int_\alpha \frac{[\omega(t)-\bar{\omega}(1/t)]^2}{t-\zeta}\,dt-\int_\beta\right]$$

$$+\frac{1+2\sigma}{24\pi i}\int\frac{[\omega(t)-\bar{\omega}(1/t)]^3}{t-\zeta}\,dt+\frac{1+\sigma}{2\pi i}\int_\gamma\frac{F(t)-\bar{F}(1/t)}{t-\zeta}\,dt, \tag{2.3}$$

$$F_2(\zeta) = \frac{\sigma a}{4\pi}\left[\int \frac{[\omega(t)-\omega(1/t)]^2}{t-\zeta}\,dt - \int\right] - \frac{\sigma}{8\pi}\left[\int_\alpha \frac{[\omega(t)+\bar{\omega}(1/t)][\omega(t)-\omega(1/t)]^2}{t-\zeta}\,dt - \int_\beta\right]$$

$$-\frac{(1+\sigma)b}{2\pi i}\int_\gamma \frac{G(t)-\bar{G}(1/t)}{t-\zeta}\,dt + \frac{1+\sigma}{8\pi}\left[\int_\alpha \frac{H(t)+H(1/t)}{t-\zeta}\,dt - \int_\beta\right] \quad (2.4)$$

where

$$F(\zeta) = \int_1^\zeta [\omega(t)-a]\omega(1/t)d\omega(t), \quad (2.5)$$

$$G(\zeta) = \int_1^\zeta \bar{\omega}(1/t)d\omega(t), \quad (2.6)$$

$$H(\zeta) = \int_1^\zeta [\omega(t)-\bar{\omega}(1/t)]^2 d\omega(t). \quad (2.7)$$

The torsional rigidity of the beam is given by (Ghosh 1947*b*, p. 108)

$$\mu N_0 = \mu(N_{01}+N_{02}) \quad (2.8)$$

where

$$N_{01} = 2\iint y^2 dxdy \quad (2.9)$$

$$N_{02} = -\tfrac{1}{2}\mathcal{R}\int_\alpha F_0(\zeta)d[\omega(\zeta)\bar{\omega}(1/\zeta)] - \tfrac{1}{2}\int_{-1}^1 F_0(\xi)d[\omega(\xi)\bar{\omega}(1/\xi)] \quad (2.10)$$

the integration in N_{01} being taken over the area of the semicardiod. The flexural moments are given by (Ghosh 1948, p. 79)

$$N_1 = N_{11}+N_{12}+N_{13}, \quad N_2 = N_{21}+N_{22}+N_{23} \quad (2.11)$$

where,

$$N_{11} \doteq \iint_R [(1+\sigma)(x-a)^2(y-b)-\sigma(y-b)^3+2(1+\sigma)axy-(1+\sigma)a(ay+bx)]dxdy, \quad (2.12)$$

$$N_{21} = \iint_R [\sigma(x-a)^3-(1+\sigma)(x-a)(y-b)^2+3\sigma a(x^2-y^2)-2(1+2\sigma)bxy$$

$$-\sigma(x^3-3xy^2)+\{(1+\sigma)b^2-2\sigma a^2\}x+(1+3\sigma)aby]dxdy, \quad (2.13)$$

$$N_{r2} = -\tfrac{1}{2}\mathcal{R}\int_\alpha F_r(\zeta)d[\omega(\zeta)\bar{\omega}(1/\zeta)] - \tfrac{1}{2}\int_{-1}^1 F_r(\xi)d[\omega(\xi)\bar{\omega}(\xi)], \quad (r = 1,\ 2) \quad (2.14)$$

$$N_{r3} = \tfrac{1}{2}\mathcal{R}(a+ib)\left[\int_\alpha F_r(\zeta)d[\bar{\omega}(1/\zeta)] + \int_{-1}^1 F_r(\xi)d\bar{\omega}(\xi)\right], \quad (r = 1,\ 2) \quad (2.15)$$

the integrations in N_{11} and N_{21} being taken over the area of the semi-cardioid.

The co-ordinates of the centre of flexure referred to axes through the centroid parallel to the coordinate axes are given by (Ghosh 1947*a*, p. 7)

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$$x_f' = \frac{I_{11}N_2 - I_{12}N_1}{2(1+\sigma)(I_{11}I_{22} - I_{12}^2)}, \quad y_f' = \frac{I_{12}N_2 - I_{22}N_1}{2(1+\sigma)(I_{11}I_{22} - I_{12}^2)} \tag{2.16}$$

where I_{11}, I_{22}, and I_{12} are respectively the moments and products of inertia of the area of the semicardioid with respect to the axes through the centroid of the cross-section parallel to the coordinate axes.

In evaluating the integrals it may be observed that the amplitude of t changes from 0 to π and from $-\pi$ to 0 respectively, as we describe α and β in the positive sense.

TORSION PROBLEM

3. We consider the torsion problem of a beam whose cross-section is a semicardiod bounded by $r = 2c(1 - \cos\theta)$ and the initial line.

The mapping function which makes a conformal representation of the complete cardiod on the unit circle and hence the semi-cardiod on the semi-circle, and the real axis, on the bounding diameter of the semi-circle, is given by (Sokolnikoff, p. 245)

$$z = \omega(\zeta) = -c(1-\zeta)^2 \tag{3.1}$$

Substituting for $\omega(t)$ in (2.1) we get

$$F_0(\zeta) = -\frac{c^2}{4\pi}\left[\int_\alpha \left(t^4 - 4t^3 + 4t^2 + 4t - 10 + \frac{4}{t} + \frac{4}{t^2} - \frac{4}{t^3} + \frac{1}{t^4}\right)\frac{dt}{t-\zeta} - \int_\beta\right].$$

Evaluating the integrals, we get

$$F_0(\zeta) = \frac{c^2}{\pi}\left[\left(\zeta^3 - 4\zeta^2 + \frac{13}{3}\zeta + \frac{13}{3\zeta} - \frac{4}{\zeta^2} + \frac{1}{\zeta^3}\right) - \frac{1}{2}\left(\zeta^4 - 4\zeta^3 + 4\zeta^2 + 4\zeta - 10 + \frac{4}{\zeta} + \frac{4}{\zeta^2} - \frac{4}{\zeta^3} + \frac{1}{\zeta^4}\right)\log\frac{1+\zeta}{1-\zeta}\right] \tag{3.2}$$

omitting a constant.

4. Before calculating the torsional rigidity, we will discuss some definite integrals which will be used in evaluating N_{03}. Let us consider the integral,

$$\int_\alpha \zeta^p \log\frac{1+\zeta}{1-\zeta}\,d\zeta,$$

where $p = 0$, or a positive or a negative integer.

The integrand has singularities at the point $\zeta = 1$, and $\zeta = -1$. But as $(1-\zeta)\times\log(1-\zeta)$ and $(1+\zeta)\log(1+\zeta)$ tend to zero as ζ tends to the points $\zeta = 1$ and $\zeta = -1$ respectively along α, the integral is convergent. Integrating by parts, we easily find that when n is zero or a positive integer,

$$\int_\alpha \zeta^{2n}\log\frac{1+\zeta}{1-\zeta}\,d\zeta = 0, \tag{4.1}$$

$$\int_a \zeta^{2n+1} \log \frac{1+\zeta}{1-\zeta} d\zeta = -\frac{2}{n+1}\left(1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n+1}\right). \tag{4.2}$$

When $n \neq 0$, we also find by the same process,

$$\int_a \log \frac{1+\zeta}{1-\zeta} \frac{d\zeta}{\zeta^{2n}} = \frac{2i\pi}{2n-1} \tag{4.3}$$

$$\int_a \log \frac{1+\zeta}{1-\zeta} . \frac{d\zeta}{\zeta^{2n+1}} = \frac{2}{n}\left(1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n-1}\right) \tag{4.4}$$

It is also seen at once that

$$\int_a \frac{1}{2\zeta} \log \frac{1+\zeta}{1-\zeta} d\zeta = -2\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right) = -\frac{\pi^2}{4}. \tag{4.5}$$

Let us next consider the integral,

$$\int_{-1}^{1} \xi^p \log \frac{1+\xi}{1-\xi} d\xi.$$

When $p = 0$ or a positive integer, the integral can be evaluated as before, and we find for $n \geq 0$

$$\int_{-1}^{1} \xi^{2n} \log \frac{1+\xi}{1-\xi} d\xi = 0, \tag{4.6}$$

$$\int_{-1}^{1} \xi^{2n+1} \log \frac{1+\xi}{1-\xi} d\xi = \frac{2}{n+1}\left(1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n+1}\right) \tag{4.7}$$

But when p is a negative integer not equal to -1, the integrand has a singularity at $\xi = 0$ in addition to those at $\xi = \pm 1$. In such cases, we consider integrals of the types

$$I_{2n} = \int_{-1}^{1}\left[\frac{1}{2\xi^{2n}} \log \frac{1+\xi}{1-\xi} - \frac{1}{1.\xi^{2n-1}} - \frac{1}{3.\xi^{2n-3}} - \frac{1}{5.\xi^{2n-5}} - \cdots - \frac{1}{(2n-1)\xi}\right] d\xi,$$

$$I_{2n+1} = \int_{-1}^{1}\left[\frac{1}{2\xi^{2n+1}} \log \frac{1+\xi}{1-\xi} - \frac{1}{1.\xi^{2n}} - \frac{1}{3.\xi^{2n-2}} - \frac{1}{5.\xi^{2n-4}} - \cdots - \frac{1}{(2n-1)\xi^2}\right] d\xi,$$

when $n \neq 0$.

Suitable terms with negative powers of ξ have been introduced to cancel the singularity at $\xi = 0$ These integrals can now be evaluated by integrating by parts, and we find,

$$I_{2n} = 0, \tag{4.8}$$

$$I_{2n+1} = \frac{1}{n}\left(1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n-1}\right). \tag{4.9}$$

Finally,

$$I_1 = \int_{-1}^{1} \frac{1}{2\xi} \log \frac{1+\xi}{1-\xi} d\xi = \frac{\pi^2}{4} \tag{4.10}$$

5. We have,

$$N_{01} = \iint y^2 dxdy = \frac{21\pi c^4}{2} \tag{5.1}$$

and

$$N_{02} = -\tfrac{1}{2}c^2 \mathcal{R}\int_a F_0(\zeta) d\left[\frac{(1-\zeta)^4}{\zeta^2}\right] - \tfrac{1}{2}c^2 \int_{-1}^{1} F_0(\xi) d[(1-\xi)^4]$$

$$= -\frac{c^4}{\pi}\mathcal{R}\int_a \left[f_{01}(\zeta) + f_{02}(\zeta)\log\frac{1+\zeta}{1-\zeta}\right] d\zeta - \frac{2c^4}{\pi}\int_{-1}^{1}\left[g_{01}(\xi) + g_{02}(\xi)\log\frac{1+\xi}{1-\xi}\right] d\xi$$

$$-\frac{2c^4}{\pi}\int_{-1}^{1}\left[g_{03}(\xi) + g_{04}(\xi)\log\frac{1+\xi}{1-\xi}\right] d\xi,$$

where,

$$f_{01}(\zeta) = \zeta^4 - 6\zeta^3 + \frac{37}{3}\zeta^2 - \frac{20}{3}\zeta - \frac{14}{3} + \frac{14}{3\zeta^2} + \frac{20}{3\zeta^3} - \frac{37}{3\zeta^4} + \frac{6}{\zeta^5} - \frac{1}{\zeta^6},$$

$$f_{02}(\zeta) = -\frac{1}{2}\left(\zeta^5 - 6\zeta^4 + 12\zeta^3 - 2\zeta^2 - 27\zeta + 36 - \frac{36}{\zeta^2} + \frac{27}{\zeta^3} + \frac{2}{\zeta^4} - \frac{12}{\zeta^5} + \frac{6}{\zeta^6} - \frac{1}{\zeta^7}\right),$$

$$g_{01}(\xi) = \xi^6 - 7\xi^5 + \frac{58}{3}\xi^4 - 26\xi^3 + \frac{64}{3}\xi^2 - \frac{64}{3}\xi + 26,$$

$$g_{02}(\xi) = -\tfrac{1}{2}(\xi^7 - 7\xi^6 + 19\xi^5 - 21\xi^4 - 6\xi^3 + 42\xi^2 - 42\xi + 6),$$

$$g_{03}(\xi) = -\frac{58}{5\xi} + \frac{7}{\xi^2} - \frac{1}{\xi^3},$$

$$g_{04}(\xi) = -\frac{1}{2}\left(\frac{21}{\xi} - \frac{19}{\xi^2} + \frac{7}{\xi^3} - \frac{1}{\xi^4}\right).$$

Using results of Art. 4, we have

$$\mathcal{R}\int_a \left[f_{01}(\zeta) + f_{02}(\zeta)\log\frac{1+\zeta}{1-\zeta}\right] d\zeta = -\frac{320}{9},$$

$$\int_{-1}^{1}\left[g_{01}(\xi) + g_{02}(\xi)\log\frac{1+\xi}{1-\xi}\right] d\xi = \frac{991}{9},$$

$$\int_{-1}^{1}\left[g_{03}(\xi) + g_{04}(\xi)\log\frac{1+\xi}{1-\xi}\right] d\xi = -21I_1 + 19I_2 - 7I_3 + I_4 = -\frac{21}{4}\pi^2 - 7.$$

Hence,

$$N_{02} = \frac{c^4}{\pi}(-171 + \tfrac{1}{3}) + \tfrac{21}{2}\pi c^4 \tag{5.2}$$

and, therefore, the torsional rigidity of the beam is,

$$\mu c^4\left(21\pi-\frac{512}{3\pi}\right) \tag{5.3}$$

FLEXURE PROBLEM

6. We now consider the flexure problem The coordinates of the centre of gravity of the cross-section are given by,

$$a=-\tfrac{5}{3}c, \quad b=\frac{32}{9\pi}c. \tag{6.1}$$

The first terms in $F_1(\zeta)$ and $F_2(\zeta)$ are $\sigma bF_0(\zeta)$ and $-\sigma aF_0(\zeta)$ respectively. The integral,

$$\int_\gamma [\omega(t)-\bar{\omega}(1/t)]^3\frac{dt}{t-\zeta}=c^3\int_\gamma (1-t)^6(1-t^2)^3\frac{dt}{t^6(t-\zeta)}$$

can be evaluated by Cauchy's Residue Theorem and its value, omitting an additive constant, is found to be

$$2\pi ic^3(-\zeta^6+6\zeta^5-12\zeta^4+2\zeta^3+27\zeta^2-36\zeta). \tag{6.2}$$

Also, substituting for $\omega(t)$, and integrating we get,

$$\int_\alpha [\omega(t)+\bar{\omega}(1/t)][\omega(t)-\bar{\omega}(1/t)]^2\frac{dt}{t-\zeta}-\int_\beta$$

$$=4c^3\left[\left(\zeta^5-6\zeta^4+\frac{43}{3}\zeta^3-16\zeta^2+\frac{58}{15}\zeta+\frac{58}{15\zeta}-\frac{16}{\zeta^2}+\frac{43}{3\zeta^3}-\frac{6}{\zeta^4}+\frac{1}{\zeta^5}\right)\right.$$

$$\left.-\left(\frac{\zeta^6}{2}-8\zeta^5+7\zeta^4-7\zeta^3-\frac{\zeta^2}{2}+10\zeta-14+\frac{10}{\zeta}-\frac{1}{2\zeta^2}-\frac{7}{\zeta^3}+\frac{7}{\zeta^4}-\frac{3}{\zeta^5}+\frac{1}{2\zeta^6}\right)\log\frac{1+\zeta}{1-\zeta}\right], \tag{6.3}$$

omitting a constant.

From (2.5), (2.6), and (2.7) we get,

$$F(\zeta)=c^3\left[\frac{-\zeta^4}{2}+\frac{10}{3}\zeta^3-\frac{25}{3}\zeta^2+10\zeta-\frac{35}{6}+\frac{4}{3\zeta}\right] \tag{6.4}$$

$$G(\zeta)=c^2\left[\zeta^2-6\zeta+3+\frac{2}{\zeta}+6\log\zeta\right] \tag{6.5}$$

$$H(\zeta)=c^3\left[-\frac{\zeta^6}{3}+2\zeta^5-4\zeta^4+14\zeta^2-28\zeta+28-\frac{16}{\zeta}+\frac{5}{\zeta^2}-\frac{2}{3\zeta^3}\right] \tag{6.6}$$

With the help of the integral

$$\int_\gamma \frac{\log t}{t-\zeta}dt=2\pi i\log(1+\zeta),$$

and using Cauchy's Residue Theorem, we have

$$\int_{\gamma} [F(t)-\bar{F}(1/t)]\,\frac{dt}{t-\zeta} = 2\pi i c^3\left(-\tfrac{1}{2}\zeta^4+\tfrac{10}{3}\zeta^3-\tfrac{25}{3}\zeta^2+\tfrac{26}{3}\zeta\right) \tag{6.7}$$

$$\int [G(t)-\bar{G}(1/t)]\,\frac{dt}{t-\zeta} = 2\pi i c^2[\zeta^2-8\zeta+12\log(1+\zeta)] \tag{6.8}$$

A direct integration gives

$$\int_{\alpha} [H(t)+\bar{H}(1/t)]\,\frac{dt}{t-\zeta} - \int_{\beta}$$

$$= 4c^3\left[\left(\frac{1}{3}\zeta^5-2\zeta^4+\frac{37}{9}\zeta^3-\frac{88}{5}\zeta-\frac{88}{5\zeta}+\frac{37}{9\zeta^3}-\frac{2}{\zeta^4}+\frac{1}{3\zeta^5}\right)\right.$$

$$\left.+\left(-\frac{1}{6}\zeta^6+\zeta^5-2\zeta^4-\frac{1}{3}\zeta^3+\frac{19}{2}\zeta^2-22\zeta+28-\frac{22}{\zeta}+\frac{19}{2\zeta^2}-\frac{1}{3\zeta^3}-\frac{2}{\zeta^4}+\frac{1}{\zeta^5}-\frac{1}{6\zeta^6}\right)\log\frac{1+\zeta}{1-\zeta}\right] \tag{6.9}$$

omitting a constant.

Using these results, we get

$$F_1(\zeta) = \sigma b F_0(\zeta)+\frac{c^3}{12}(-\zeta^6+6\zeta^5-18\zeta^4+42\zeta^3-72\zeta^2+68\zeta)$$

$$+\frac{\sigma c^3}{6}(-\zeta^6+6\zeta^5-15\zeta^4+22\zeta^3-23\zeta^2+16\zeta), \tag{6.10}$$

$$F_2(\zeta) = -\sigma a F_0(\zeta)-\frac{\sigma c^3}{2\pi}\left[\left(\zeta^5-6\zeta^4+\frac{43}{3}\zeta^3-16\zeta^2+\frac{58}{15}\zeta+\frac{58}{15\zeta}-\frac{16}{\zeta^2}+\frac{43}{3\zeta^3}-\frac{6}{\zeta^4}+\frac{1}{\zeta^5}\right)\right.$$

$$\left.-\left(\frac{\zeta^6}{2}-3\zeta^5+7\zeta^4-7\zeta^3-\frac{\zeta^2}{2}+10\zeta-14+\frac{10}{\zeta}-\frac{1}{2\zeta^2}-\frac{7}{\zeta^3}+\frac{7}{\zeta^4}-\frac{3}{\zeta^5}+\frac{1}{2\zeta^6}\right)\log\frac{1+\zeta}{1-\zeta}\right]$$

$$+\frac{(1+\sigma)c^3}{6\pi}\left[\zeta^5-6\zeta^4+\frac{37}{3}\zeta^3-\frac{64}{3}\zeta^2+\frac{1768}{15}\zeta-\frac{264}{5\zeta}+\frac{37}{3\zeta^3}-\frac{6}{\zeta^4}+\frac{1}{\zeta^5}\right]$$

$$+\frac{(1+\sigma)c^3}{2\pi}\left[\left(-\frac{\zeta^6}{6}+\zeta^5-2\zeta^4-\frac{\zeta^3}{3}+\frac{19}{2}\zeta^2-22\zeta+28-\frac{22}{\zeta}+\frac{19}{2\zeta^2}-\frac{1}{3\zeta^3}\right.\right.$$

$$\left.\left.-\frac{2}{\zeta^4}+\frac{1}{\zeta^5}-\frac{1}{6\zeta^6}\right)\log\frac{1+\zeta}{1-\zeta}-\frac{256}{3}\log(1+\zeta)\right] \tag{6.11}$$

7. Integrating over a cross-section, we get

$$\iint dxdy = 3\pi c^2, \qquad \iint xy\,dxdy = -\frac{256}{15}c^4, \tag{7.1}$$

$$\iint x^2 dxdy = \frac{49}{4}\pi c^4, \qquad \iint y^2 dxdy = \frac{21}{4}\pi c^4, \tag{7.2}$$

$$\iint x^2 y\,dx\,dy = \frac{4096}{105}c^5, \qquad \iint y^3\,dx\,dy = \frac{1024}{35}c^5, \qquad \iint xy^2\,dx\,dy = -\frac{33\pi c^5}{4} \tag{7.3}$$

Substituting in (2.12) we get.

$$N_{11} = c^5\left[-\left(4+\frac{172}{315}\right)+\sigma\left\{22+\frac{62}{315}-\frac{2(32)^3}{243\pi^2}\right\}\right] \tag{7.4}$$

Substituting for $F_1(\zeta)$ from (6 10) in (2.14) we find

$$\begin{aligned} N_{12} = \sigma b N_{02} &- \frac{c^5}{12}\mathcal{R}\int_a \left(-\zeta^7+8\zeta^6-30\zeta^5+76\zeta^4-144\zeta^3+172\zeta^2-34\zeta-188+\frac{209}{\zeta}-\frac{68}{\zeta^2}\right)d\zeta \\ &- \frac{c^5\sigma}{6}\mathcal{R}\int_a \left(-\zeta^7+8\zeta^6-27\zeta^5+50\zeta^4-54\zeta^3+26\zeta^2+27\zeta-68+\frac{55}{\zeta}-\frac{16}{\zeta^3}\right)d\zeta \\ &- \frac{c^5}{6}\int_{-1}^{1}(-\xi^9+9\xi^8-39\xi^7+115\xi^6-259\xi^5+431\xi^4-465\xi^3+277\xi^2-68\xi)\,d\xi \\ &- \frac{c^5\sigma}{3}\int_{-1}^{1}(-\xi^9+9\xi^8-36\xi^7+86\xi^6-140\xi^5+166\xi^4-139\xi^3+71\xi^2-16\xi)\,d\xi. \end{aligned}$$

Substituting the value of b we get

$$N_{12} = -c^5\left[\left(73+\frac{13}{315}\right)+\sigma\left(20+\frac{52}{315}\right)+\frac{16\sigma}{9\pi^2}\left(341+\frac{1}{3}\right)\right] \tag{7 5}$$

Substituting for $F_1(\zeta)$, we have

$$\begin{aligned} &\int_a F_1(\zeta)\,d[\omega(1/\zeta)] + \int_{-1}^{1} F_1(\xi)\,d\bar{\omega}(\xi) \\ &= -\frac{2\sigma b c^3}{\pi}\int_a \left\{f_{11}(\zeta)+f_{12}(\zeta)\log\frac{1+\zeta}{1-\zeta}\right\}d\zeta + \frac{c^4}{6}\int_a f_{13}(\zeta)\,d\zeta + \frac{c^4\sigma}{3}\int_a f_{14}(\zeta)\,d\zeta \\ &\quad -\frac{2\sigma b c^3}{\pi}\int_{-1}^{1}\left\{g_{11}(\xi)+g_{12}(\xi)\log\frac{1+\xi}{1-\xi}\right\}d\xi - \frac{2\sigma b c^3}{\pi}\int_{-1}^{1}\left\{g_{13}(\xi)+g_{14}(\xi)\log\frac{1+\xi}{1-\xi}\right\}d\xi \\ &\quad + \frac{c^4}{6}\int_{-1}^{1} g_{15}(\xi)\,d\xi + \frac{c^4\sigma}{3}\int_{-1}^{1} g_{16}(\xi)\,d\xi, \end{aligned}$$

where

$$f_{11}(\zeta) = \zeta-5+\frac{25}{3\zeta}-\frac{13}{3\zeta^2}+\frac{13}{3\zeta^3}-\frac{25}{3\zeta^4}+\frac{5}{\zeta^5}-\frac{1}{\zeta^6},$$

$$f_{12}(\zeta) = -\frac{1}{2}\zeta^2+\frac{5}{2}\zeta-4+\frac{7}{\zeta^2}-\frac{7}{\zeta^3}+\frac{4}{\zeta^5}-\frac{5}{2\zeta^6}+\frac{1}{2\zeta^7},$$

$$f_{13}(\zeta) = \zeta^4-7\zeta^3+24\zeta^2-60\zeta+115-\frac{141}{\zeta}+\frac{68}{\zeta^2},$$

$$f_{14}(\zeta) = \zeta^4 - 7\zeta^3 + 21\zeta^2 - 37\zeta + 45 - \frac{39}{\zeta} + \frac{16}{\zeta^2},$$

$$g_{11}(\xi) = \xi^4 - 5\xi^3 + \tfrac{25}{3}\xi^2 - \tfrac{13}{3}\xi + \tfrac{13}{3}, \qquad g_{12}(\xi) = -\tfrac{1}{2}\xi^5 + \tfrac{5}{2}\xi^4 - 4\xi^3 + 7\xi - 7,$$

$$g_{13}(\xi) = -\frac{25}{3\xi} + \frac{5}{\xi^2} - \frac{1}{\xi^3}, \qquad g_{14}(\xi) = \frac{4}{\xi^2} - \frac{5}{2\xi^3} + \frac{1}{2\xi^4},$$

$$g_{15}(\xi) = \xi^7 - 7\xi^6 + 24\xi^5 - 60\xi^4 + 115\xi^3 - 141\xi^2 + 68\xi,$$

$$g_{16}(\xi) = \xi^7 - 7\xi^6 + 21\xi^5 - 37\xi^4 + 45\xi^3 - 39\xi^2 + 16\xi.$$

Using results of § 4, we have

$$\int_a \left\{ f_{11}(\zeta) + f_{12}(\zeta) \log \frac{1+\zeta}{1-\zeta} \right\} d\zeta = -\left(\frac{160}{9} - \frac{64}{3}\pi i \right),$$

$$\int_a f_{13}(\zeta) d\zeta = -\frac{552}{5} - 141\pi i, \qquad \int_a f_{14}(\zeta) d\zeta = -\frac{362}{5} - 39\pi i,$$

$$\int_{-1}^{1} \left\{ g_{11}(\xi) + g_{12}(\xi) \log \frac{1+\xi}{1-\xi} \right\} d\xi = \frac{205}{9},$$

$$\int_{-1}^{1} \left\{ g_{13}(\xi) + g_{14}(\xi) \log \frac{1+\xi}{1-\xi} \right\} d\xi = 8I_2 - 5I_3 + I_1 = -5,$$

$$\int_{-1}^{1} g_{15}(\xi) d\xi = -120, \qquad \int_{-1}^{1} g_{16}(\xi) d\xi = -\frac{214}{5}.$$

Substituting in (2.15),

$$N_{13} = c^5 \left[\frac{664}{9} + \sigma \left(\frac{496}{9} + \frac{65536}{243\pi^2} \right) \right] \tag{7.6}$$

Hence

$$N_1 = c^5 \left[-\frac{80}{21} - \frac{16384}{27\pi^2}\sigma + \frac{400}{7}\sigma \right]. \tag{7.7}$$

8. Substituting from (7.1), (7.2), (7.3) in (2.13), we get

$$N_{21} = c^5 \left[-\frac{\pi}{2} + \sigma \left(\pi - \frac{2048}{405\pi} \right) \right] \tag{8.1}$$

Again from (2.14) and (6.11) we get

$$N_{22} = -\sigma a N_{02} + \frac{\sigma c^5}{2\pi} \mathcal{R} \int_a \left\{ f_{21}(\zeta) + f_{22}(\zeta) \log \frac{1+\zeta}{1-\zeta} \right\} d\zeta$$

$$-\frac{(1+\sigma)c^5}{6\pi} \mathcal{R} \int_a \left\{ f_{23}(\zeta) + f_{24}(\zeta) \log \frac{1+\zeta}{1-\zeta} + f_{25}(\zeta) \log(1+\zeta) \right\} d\zeta$$

$$+\frac{\sigma c^5}{\pi}\int_{-1}^{1}\left\{g_{21}(\xi)+g_{22}(\xi)\log\frac{1+\xi}{1-\xi}\right\}d\xi+\frac{\sigma c^5}{\pi}\int_{-1}^{1}\left\{g_{23}(\xi)+g_{24}(\xi)\log\frac{1+\xi}{1-\xi}\right\}d\xi$$

$$-\frac{(1+\sigma)c^5}{3\pi}\int_{-1}^{1}\left\{g_{25}(\xi)+g_{26}(\xi)\log\frac{1+\xi}{1-\xi}+g_{27}(\xi)\log(1+\xi)\right\}d\xi$$

$$-\frac{(1+\sigma)c^5}{3\pi}\int_{-1}^{1}\left\{g_{28}(\xi)+g_{29}(\xi)\log\frac{1+\xi}{1-\xi}\right\}d\xi \qquad (8.2)$$

where

$$f_{21}(\zeta)=\zeta^6-8\zeta^5+\frac{79}{3}\zeta^4-\frac{128}{3}\zeta^3+\frac{343}{15}\zeta^2+\frac{404}{15}\zeta-\frac{637}{15}$$

$$+\frac{637}{15\zeta^2}-\frac{404}{15\zeta^3}-\frac{343}{15\zeta^4}+\frac{128}{3\zeta^5}-\frac{79}{3\zeta^6}+\frac{8}{\zeta^7}-\frac{1}{\zeta^8},$$

$$f_{22}(\zeta)=-\frac{\zeta^7}{2}+4\zeta^6-13\zeta^5+20\zeta^4-7\zeta^3-28\zeta^2+55\zeta-44$$

$$+\frac{44}{\zeta^2}-\frac{55}{\zeta^3}+\frac{28}{\zeta^4}+\frac{7}{\zeta^5}-\frac{20}{\zeta^6}+\frac{13}{\zeta^7}-\frac{4}{\zeta^8}+\frac{1}{2\zeta^9},$$

$$f_{23}(\zeta)=\zeta^6-8\zeta^5+\frac{73}{3}\zeta^4-\frac{132}{3}\zeta^3+\frac{2213}{15}\zeta^2-\frac{3076}{15}\zeta-\frac{1617}{15}$$

$$+\frac{1088}{3\zeta}-\frac{1583}{15\zeta^2}-\frac{2044}{15\zeta^3}+\frac{329}{5\zeta^4}+\frac{68}{3\zeta^5}-\frac{73}{3\zeta^6}+\frac{8}{\zeta^7}-\frac{1}{\zeta^8},$$

$$f_{24}(\zeta)=-\frac{\zeta^7}{2}+4\zeta^6-12\zeta^5+10\zeta^4+37\zeta^3-138\zeta^2+220\zeta-176$$

$$+\frac{176}{\zeta^2}-\frac{220}{\zeta^3}+\frac{138}{\zeta^4}-\frac{37}{\zeta^5}-\frac{10}{\zeta^6}+\frac{12}{\zeta^7}-\frac{4}{\zeta^8}+\frac{1}{2\zeta^9},$$

$$f_{25}(\zeta)=-256\left(\zeta-2+\frac{2}{\zeta^2}-\frac{1}{\zeta^3}\right),$$

$$g_{21}(\xi)=\xi^8-9\xi^7+\frac{106}{3}\xi^6-78\xi^5+\frac{1513}{15}\xi^4-\frac{1109}{15}\xi^3+\frac{472}{15}\xi^2-\frac{472}{15}\xi+\frac{1109}{15},$$

$$g_{22}(\xi)=-\tfrac{1}{2}(\xi^9-9\xi^8+35\xi^7-75\xi^6+89\xi^5-33\xi^4-77\xi^3+165\xi^2-165\xi+77),$$

$$g_{23}(\xi)=-\frac{1513}{15\xi}+\frac{78}{\xi^7}-\frac{106}{3\xi^3}+\frac{9}{\xi^4}-\frac{1}{\xi^5},$$

$$g_{24}(\xi)=-\frac{1}{2}\left(\frac{33}{\xi}-\frac{89}{\xi^2}+\frac{75}{\xi^3}-\frac{35}{\xi^4}+\frac{9}{\xi^5}-\frac{1}{\xi^6}\right),$$

$$g_{25}(\xi) = \xi^8 - 9\xi^7 + \frac{100}{3}\xi^6 - \frac{232}{3}\xi^5 + \frac{3373}{15}\xi^4 - \frac{6449}{15}\xi^3 + \frac{4832}{15}\xi^2 + \frac{608}{15}\xi - \frac{2191}{15},$$

$$g_{26}(\xi) = \tfrac{1}{2}(-\xi^9 + 9\xi^8 - 33\xi^7 + 53\xi^6 + 21\xi^5 - 297\xi^4 + 737\xi^3 - 1089\xi^2 + 1089\xi - 737),$$

$$g_{27}(\xi) = -256(\xi^3 - 3\xi^2 + 3\zeta - 1), \qquad g_{28}(\xi) = \frac{49}{5\xi} + \frac{56}{\xi^2} - \frac{100}{3\xi^3} + \frac{9}{\xi^4} - \frac{1}{\xi^5},$$

$$g_{29}(\xi) = \frac{1}{2}\left(\frac{297}{\xi} - \frac{21}{\xi^2} - \frac{53}{\xi^3} + \frac{33}{\xi^4} - \frac{9}{\xi^5} + \frac{1}{\xi^6}\right).$$

Using results of § 4, we have

$$\mathcal{R}\int_a \left\{f_{21}(\zeta) + f_{22}(\zeta)\log\frac{1+\zeta}{1-\zeta}\right\} d\zeta = -\frac{2528}{45},$$

$$\mathcal{R}\int_a \left\{f_{23}(\zeta) + f_{24}(\zeta)\log\frac{1+\zeta}{1-\zeta} + f_{25}(\zeta)\log(1+\zeta)\right\} d\zeta = 512 + \frac{32}{45} - 2048\log 2,$$

$$\int_{-1}^{1}\left\{g_{21}(\xi) + g_{22}(\xi)\log\frac{1+\xi}{1-\xi}\right\} d\xi = 375 + \frac{74}{225},$$

$$\int_{-1}^{1}\left\{g_{23}(\xi) + g_{24}(\xi)\log\frac{1+\xi}{1-\xi}\right\} d\xi = -83I_1 + 80I_2 - 75I_3 + 35I_4 - 9I_5 + I_6 = -81 - \frac{33}{4}\pi^2,$$

$$\int_{-1}^{1}\left\{g_{25}(\xi) + g_{26}(\xi)\log\frac{1+\xi}{1-\xi} + g_{27}(\xi)\log(1+\xi)\right\} d\xi = -534 - \frac{31}{225} + 1024\log 2,$$

$$\int_{-1}^{1}\left\{g_{28}(\xi) + g_{29}(\xi)\log\frac{1+\xi}{1-\xi}\right\} d\xi = 297I_1 - 21I_2 - 53I_3 + 33I_4 - 9I_5 + I_6 = -59 + \frac{297}{4}\pi^2$$

Substituting in (8.2), we get

$$N_{22} = \frac{c^5}{\pi}\left[112 + \frac{176}{675} - \frac{99}{4}\pi^2 + \sigma\left(94 + \frac{38}{675} - \frac{31}{2}\pi^2\right)\right]. \tag{8.3}$$

Again from (6.11), we have

$$\int_a F_2(\zeta)d\omega(1/\zeta) + \int_{-1}^{1} F_2(\xi)d\bar{\omega}(\xi)$$

$$= \frac{2\sigma ac^3}{\pi}\int_a\left\{f_{11}(\zeta) + f_{12}(\zeta)\log\frac{1+\zeta}{1-\zeta}\right\}d\zeta - \frac{\sigma c^4}{\pi}\int_a\left\{f_{31}(\zeta) + f_{32}(\zeta)\log\frac{1+\zeta}{1-\zeta}\right\}d\zeta$$

$$+ \frac{(1+\sigma)c^4}{3\pi}\int_a\left\{f_{33}(\zeta) + f_{34}(\zeta)\log\frac{1+\zeta}{1-\zeta} + f_{35}(\zeta)\log(1+\zeta)\right\}d\zeta$$

$$+ \frac{2\sigma ac^3}{\pi}\left[\int_{-1}^{1}\left\{g_{11}(\xi) + g_{12}(\xi)\log\frac{1+\xi}{1-\xi}\right\}d\xi + \int_{-1}^{1}\left\{g_{13}(\xi) + g_{14}(\xi)\log\frac{1+\xi}{1-\xi}\right\}d\xi\right]$$

$$-\frac{\sigma c^4}{\pi}\left[\int_{-1}^{1}\left\{g_{31}(\xi)+g_{32}(\xi)\log\frac{1+\xi}{1-\xi}\right\}d\xi+\int_{-1}^{1}\left\{g_{33}(\xi)+g_{34}(\xi)\log\frac{1+\xi}{1-\xi}\right\}d\xi\right]$$

$$+\frac{(1+\sigma)c^4}{3\pi}\left[\int_{-1}^{1}\left\{g_{35}(\xi)+g_{36}(\xi)\log\frac{1+\xi}{1-\xi}+g_{37}(\xi)\log(1+\xi)\right\}d\xi\right.$$

$$\left.+\int_{-1}^{1}\left\{g_{38}(\xi)+g_{39}(\xi)\log\frac{1+\xi}{1-\xi}\right\}d\xi\right] \quad (8.4)$$

where

$$f_{31}(\zeta) = -\zeta^3+7\zeta^2-\frac{61}{3}\zeta+\frac{91}{3}-\frac{298}{15\zeta}+\frac{58}{15\zeta^2}-\frac{58}{15\zeta^3}+\frac{298}{15\zeta^4}-\frac{91}{3\zeta^5}+\frac{61}{3\zeta^6}-\frac{7}{\zeta^7}+\frac{1}{\zeta^8},$$

$$f_{32}(\zeta) = -\frac{1}{2}\left(-\zeta^4+7\zeta^3-20\zeta^2+28\zeta-13-\frac{21}{\zeta}+\frac{48}{\zeta^2}-\frac{48}{\zeta^3}+\frac{21}{\zeta^4}+\frac{13}{\zeta^5}-\frac{28}{\zeta^6}+\frac{20}{\zeta^7}-\frac{7}{\zeta^8}+\frac{1}{\zeta^9}\right),$$

$$f_{33}(\zeta) = -\zeta^3+7\zeta^2-\frac{55}{3}\zeta+\frac{101}{3}-\frac{696}{5\zeta}+\frac{1768}{15\zeta^2}+\frac{264}{5\zeta^3}-\frac{264}{5\zeta^4}-\frac{37}{3\zeta^5}+\frac{55}{3\zeta^6}-\frac{7}{\zeta^7}+\frac{1}{\zeta^8},$$

$$f_{34}(\zeta) = \frac{\zeta^4}{2}-\frac{7}{2}\zeta^3+9\zeta^2-5\zeta-\frac{59}{2}+\frac{189}{2\zeta}-\frac{150}{\zeta^2}+\frac{150}{\zeta^3}-\frac{189}{2\zeta^4}+\frac{59}{2\zeta^5}+\frac{5}{\zeta^6}-\frac{9}{\zeta^7}+\frac{7}{2\zeta^8}-\frac{1}{2\zeta^9},$$

$$f_{35}(\zeta) = -256\left(\frac{1}{\zeta^3}-\frac{1}{\zeta^2}\right),$$

$$g_{31}(\xi) = -\xi^6+7\xi^5-\frac{61}{3}\xi^4+\frac{91}{3}\xi^3-\frac{298}{15}\xi^2+\frac{58}{15}\xi-\frac{58}{15},$$

$$g_{32}(\xi) = \frac{\xi^7}{2}-\frac{7}{2}\xi^6+10\xi^5-14\xi^4+\frac{13}{2}\xi^3+\frac{21}{2}\xi^2-24\xi+24,$$

$$g_{33}(\xi) = \frac{298}{15\xi}-\frac{91}{3\xi^2}+\frac{61}{3\xi^3}-\frac{7}{\xi^4}+\frac{1}{\xi^5},$$

$$g_{34}(\xi) = -\frac{21}{2\xi}-\frac{13}{2\xi^2}+\frac{14}{\xi^3}-\frac{10}{\xi^4}+\frac{7}{2\xi^5}-\frac{1}{2\xi^6},$$

$$g_{35}(\xi) = -\xi^6+7\xi^5-\frac{55}{3}\xi^4+\frac{101}{3}\xi^3-\frac{696}{5}\xi^2+\frac{1768}{15}\xi+\frac{264}{5},$$

$$g_{36}(\xi) = \frac{\xi^7}{2}-\frac{7}{2}\xi^6+9\xi^5-5\xi^4-\frac{59}{2}\xi^3+\frac{189}{2}\xi^2-150\xi+150,$$

$$g_{37}(\xi) = 256(\xi-1), \qquad g_{38}(\xi) = -\frac{264}{5\xi}-\frac{37}{3\xi^2}+\frac{55}{3\xi^3}-\frac{7}{\xi^4}+\frac{1}{\xi^5},$$

$$g_{39}(\xi) = -\frac{189}{2\xi}+\frac{59}{2\xi^2}+\frac{5}{\xi^3}-\frac{9}{\xi^4}+\frac{7}{2\xi^5}-\frac{1}{2\xi^6}.$$

Using results of Art. 4, we have

$$\int_{a}\left\{f_{31}(\zeta)+f_{32}(\zeta)\log\frac{1+\zeta}{1-\zeta}\right\}d\zeta = 26-\frac{182}{315}-\frac{1024}{15}\pi i-\frac{21}{4}\pi^2,$$

$$\int_{a}\left\{f_{33}(\zeta)+f_{34}(\zeta)\log\frac{1+\zeta}{1-\zeta}+f_{35}(\zeta)\log(1+\zeta)\right\}d\zeta = 225-\frac{7}{15}-\frac{576}{5}\pi i-\frac{189}{4}\pi^2+512\log 2,$$

$$\int_{-1}^{1}\left\{g_{31}(\xi)+g_{32}(\xi)\log\frac{1+\xi}{1-\xi}\right\}d\xi = -58-\frac{4}{45},$$

$$\int_{-1}^{1}\left\{g_{33}(\xi)+g_{34}(\xi)\log\frac{1+\xi}{1-\xi}\right\}d\xi = -21I_1-13I_2+28I_3-20I_4+7I_5-I_6 = \frac{98}{3}-\frac{21}{4}\pi^2,$$

$$\int_{-1}^{1}\left\{g_{35}(\xi)+g_{36}(\xi)\log\frac{1+\xi}{1-\xi}+g_{37}(\xi)\log(1+\xi)\right\}d\xi = 444-\frac{8}{15}-512\log 2,$$

$$\int_{-1}^{1}\left\{g_{38}(\xi)+g_{39}(\xi)\log\frac{1+\xi}{1-\xi}\right\}d\xi = -189I_1+59I_2+10I_3-18I_4+7I_5-I_6 = \frac{44}{3}-\frac{189}{4}\pi^2.$$

Substituting the above results in (2.15) and using results of § 7, we have

$$N_{23} = \frac{c^5}{\pi}\left[-122+\frac{86}{135}+\frac{105}{4}\pi^2+\sigma\left(-116-\frac{124}{405}+\frac{35}{2}\pi^2\right)\right] \tag{8.5}$$

Hence from (8.1), (8.3), (8.5) and (2.11), we get

$$N_2 = \frac{c^5}{\pi}\left(\pi^2-9-\frac{23}{225}\right)(1+3\sigma) \tag{8.6}$$

Substituting from (7 7) and (8.6) in (2.16) and using the values,

$$I_{11} = \frac{47}{12}\pi c^4, \quad I_{12} = \frac{32}{45}c^4, \quad I_{22} = \left(\frac{21}{4}\pi-\frac{1024}{27\pi}\right)c^4,$$

the co-ordinates of the centre of flexure are easily obtained.

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PARALLEL DISPLACEMENT AND SCALAR PRODUCT OF VECTORS—II

By

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1. This is a continuation of a previous paper (Sen, 1948) in which an attempt was made to obtain information regarding parallel displacement of vectors in its relationship to the change in the scalar product of vectors in a Riemannian space of n dimensions with the metric

$$ds^2 = g_{ij}dx^i dx^j.$$

It was seen that corresponding to every arbitrary parallel displacement defined by

$$dV^l + \Gamma^l_{pq} V^p dx^q = 0, \tag{1.1}$$

there exists the parallel displacement defined by

$$dV^l + \bar{\Gamma}^l_{pq} V^p dx^q = 0 \tag{1.2}$$

with the property that, if the covariant derivatives with respect to (1.1) and (1.2) are denoted respectively by a comma and a semi-colon followed by indices, then

$$\bar{\Gamma}^l_{pq} = \Gamma^l_{pq} + g^{lt} g_{pt,q} \qquad \therefore \quad g_{pq,r} + g_{pq;r} = 0 \tag{1.3}$$

and therefore the two parallel displacements (1.1), (1.2) may also be defined respectively by

$$dV_i = \bar{\Gamma}^t_{iq} V_t dx^q, \quad dV_i = \Gamma^t_{iq} V_t dx^q. \tag{1.3$'$}$$

The displacements (1.1) and (1-2) were called associated or *associate*. They are identical only when the scalar product of two vectors remains invariant when the vectors are transported by either of the parallelism.

Now corresponding to (1.1) there also exists the displacement

$$dV^l + \Gamma'^l_{pq} V^p dx^q = 0, \quad \Gamma'^l_{pq} = \Gamma^l_{qp}. \tag{1.4}$$

The two parallel displacements (1.1) and (1.4) may be called *conjugate*. Obviously (1.1) is self-conjugate if Γ^l_{pq} is symmetric in p, q.

A connection between these displacements may be obtained by introducing an orthogonal ennuple at each point of the space defined by

$$g_{ij} = \sum_T {}^T h_i \, {}^T h_j, \quad {}_T h^i = g^{ij} \, {}^T h_j. \tag{1.5}$$

Analogous to Ricci's coefficients of rotation (Levi-Civita, 1927), we may form coefficients of rotations with respect to these displacements. If the covariant derivative with respect to (1.4) is denoted by a solidus followed by an index, and the coefficients of rotations with respect to (1.1), (1.2), (1.4) are defined by

$$\phi_{PQR} = {}^P h_{i,j}\,{}_Q h^i\,{}_R h^j, \quad \psi_{PQR} = {}^P h_{i;j}\,{}_Q h^i\,{}_R h^j, \quad \chi_{PQR} = {}^P h_{i|j}\,{}_Q h^i\,{}_R h^j \tag{1.6}$$

it may be seen from (1.5), (1.3) and the two conjugates that

$$2\phi_{PQR} - \phi_{QRP} = \psi_{PQR}, \quad \phi_{PQR} + \phi_{PRQ} = \chi_{PQR} + \chi_{PRQ} \tag{1.6$'$}$$

As a matter of fact, if the coefficients λ_{PQR} refer to an affine connection L^l_{pq} and

$$\Gamma^l_{pq} - L^l_{pq} = T^l_{pq}$$

then

$$\phi_{PQR} - \phi_{PRQ} = \lambda_{PQR} - \lambda_{PRQ}, \quad \text{or} \quad \phi_{PQR} + \phi_{PRQ} = \lambda_{PQR} + \lambda_{PRQ}$$

according as T^l_{pq} is symmetric or skew-symmetric in p,q.

2. In the paper referred to it was seen that if

$$dV^l + \nabla^l_{pq} V^p dx^q = 0, \quad \nabla^l_{pq} = \tfrac{1}{2}(\Gamma^l_{pq} + \Gamma^l_{qp}), \tag{2.1}$$

is an arbitrary parallel displacement with symmetric connection, and if the covariant derivative with respect to (2.1) is denoted by an ordinary bracket followed by an index, then

$$\nabla^l_{pq} = \left\{ {l \atop pq} \right\} + \tfrac{1}{2} g^{lt}\{(g_{pq})_t - (g_{pt})_q - (g_{qt})_p\}, \tag{2.1$'$}$$

where $\left\{ {l \atop pq} \right\}$ is the Christoffel symbol. As a matter of fact, (2.1′) follows directly from the covariant derivative of g_{pq} with respect to (2.1).

Now, by (1.3), the associate of (2.1) is

$$dV^l + \bar{\nabla}^l_{pq} V^p dx^q = 0, \tag{2.2}$$

where

$$\bar{\nabla}^l_{pq} = \nabla^l_{pq} + g^{lt}(g_{pt})_q = \left\{ {l \atop pq} \right\} + \tfrac{1}{2} g^{lt}\{(g_{pq})_t + (g_{pt})_q - (g_{qt})_p\} \tag{2.2$'$}$$

The parallel displacement formed with the symmetric part of $\bar{\nabla}^l_{pq}$ is

$$dV^l + \Delta^l_{pq} V^p dx^q = 0, \tag{2.3}$$

where

$$\Delta^l_{pq} = \tfrac{1}{2}(\bar{\nabla}^l_{pq} + \bar{\nabla}^l_{qp}) = \left\{ {l \atop pq} \right\} + \tfrac{1}{2} g^{lt}(g_{pq})_t \tag{2.3$'$}$$

Since, by (1.3), the covariant derivative of g_{pq} with respect to (2.2) is the negative of the same derivative with respect to (2.1), and

$$\bar{\nabla}^l_{pq} - \bar{\nabla}^l_{qp} = g^{lt}\{(g_{pt})_q - (g_{qt})_p\}$$

so, if a square bracket followed by an index denotes the covariant derivative with respect to (2.3), it may be seen that

$$[g_{pq}]_t = -\tfrac{1}{2}\{(g_{pt})_q + (g_{qt})_p\}. \tag{2.3$''$}$$

The associate of (2.3) is the parallel displacement

$$dV^l + \bar{\Delta}^l_{pq} V^p dx^q = 0 \tag{2.4}$$

where, by (2.3″),

$$\bar{\Delta}^l_{pq} = \Delta^l_{pq} + g^{lt}[g_{pt}]_q = \left\{ {l \atop pq} \right\} - \tfrac{1}{2} g^{lt}(g_{qt})_p. \tag{2.4$'$}$$

The parallelism formed with the symmetric part of $\overline{\Delta}^l_{pq}$ is

$$dV^l + \Omega^l_{pq} V^p dx^q = 0 \tag{2.5}$$

where

$$\Omega^l_{pq} = \tfrac{1}{2}(\overline{\Delta}^l_{pq} + \overline{\Delta}^l_{qp}) = \left\{ {l \atop pq} \right\} - \tfrac{1}{2} g^{lt}\{(g_{pt})_q + (g_{qt})_p\}. \tag{2.5'}$$

If a rectangle followed by an index is used to denote the covariant derivative with respect to (2.5), then

$$\boxed{g_{pq}}_t = \tfrac{1}{2}(g_{pq})_t + \tfrac{1}{4}\{(g_{pt})_q + (g_{qt})_p\}. \tag{2.5''}$$

Again, the conjugate of (2.2) is

$$dV^l + \overline{\overline{\nabla}}'^l_{pq} V^p dx^q = 0 \tag{2.6}$$

where

$$\overline{\overline{\nabla}}'^l_{pq} = \overline{\nabla}^l_{qp} = \left\{ {l \atop pq} \right\} + \tfrac{1}{2} g^{lt}\{(g_{pq})_t + (g_{qt})_p - (g_{pt})_q\}. \tag{2$\cdot$6'}$$

The conjugate of the associate and the associate of the conjugate are not, in general, the same.

If the covariant derivative with respect to (2.6) is denoted by a solidus followed by an index, it may be seen, as before, that

$$g_{pq/t} = (g_{pq})_t - (g_{pt})_q - (g_{qt})_p. \tag{2.6''}$$

From the results obtained above, we deduce the following formula:

$$\left\{ {l \atop pq} \right\} = \Omega^l_{pq} + \tfrac{1}{2}(\Delta^l_{pq} - \nabla^l_{pq}). \tag{2.7}$$

This expresses the Christoffel symbol in terms of the affine connection ∇^l_{pq} with which we started and certain others that followed. The affine connections appearing in (2.7) are all symmetric. Thus, the Levi-Civita parallelism has been expressed as a combination of other parallel displacements with symmetric connections.

Since we started with (2.1), it appears worthwhile to study this parallelism. A connection between it and the Levi-Civita parallelism can be obtained through the introduction of the coefficients of rotations. If the crooked bracket followed by an index is used to denote covariant derivative with respect to Levi-Civita parallelism, then, as in (1.6), Ricci's coefficients of rotation and those with respect to (2.1) are given respectively by

$$\gamma_{PQR} = \{{}^P h_i\}_j {}_Q h^i {}_R h^j, \quad \beta_{PQR} = ({}^P h_i)_j {}_Q h^i {}_R h^j.$$

It follows from (1.5) and (2.1') that

$$\gamma_{PQR} = \tfrac{1}{2}\{\beta_{PQR} + \beta_{QRP} + \beta_{RQP} - \beta_{PRQ} - \beta_{QPR} - \beta_{RPQ}\} \tag{2.8}$$

3. In view of the results of the preceding article we are in a position to give a connection, in terms of the rates of change in the scalar product of vectors, between Levi-Civita parallelism and the parallelism (2.1).

From (2.1'), (2.3'') and (2.6'') we have

$$\begin{Bmatrix} l \\ pq \end{Bmatrix} = \nabla^l_{pq} - \tfrac{1}{2} g^{lt}(g_{pq})_t - g^{lt}[g_{pq}]_t = \nabla^l_{pq} - \tfrac{1}{2} g^{lt} g_{pq/t}$$

Also,

$$\begin{Bmatrix} t \\ lj \end{Bmatrix} = \{\nabla^t_{lj} + g^{st}(g_{ls})_j\} + \tfrac{1}{2} g^{st}(g_{js})_l + g^{st}[g_{js}]_l = \{\nabla^t_{lj} + g^{st}(g_{ls})_j\} + \tfrac{1}{2} g^{st} g_{js/l}.$$

Put

$$u_T = (g_{pq})_t V^p dx^q {}_T h^t, \quad v_T = [g_{pq}]_t V^p dx^q {}_T h^t, \quad w_T = q_{pq/t} V^p dx^q {}_T h^t$$

Then

$$u^l = \sum_T {}_T h^l u_T, \quad v^l = \sum_T {}_T h^l v_T, \quad w^l = \sum_T {}_T h^l w_T,$$

$$u_l = \sum_T {}^T h_l u_T, \quad v_l = \sum_T {}^T h_l v_T, \quad w_l = \sum_T {}^T h_l w_T.$$

Therefore, if the notations δ and d are used to denote increments with respect to Levi-Civita parallelism and the parallelism (2.1), it follows from (2.1) and (1.3′) that

$$\left.\begin{aligned} \delta V^l &= dV^l + \tfrac{1}{2}u^l + v^l = dV^l + \tfrac{1}{2}w^l \\ \delta V_l &= dV_l + \tfrac{1}{2}u_l + v_l = dV_l + \tfrac{1}{2}w_l \end{aligned}\right\} \tag{3.1}$$

The geometrical meaning of (3.1) is obtained by observing that the vectors u_T, v_T and w_T measure the rates of change in the scalar product of the vectors V and dx when these vectors are given the parallel displacements (2.1), (2.3) and (2.6) respectively in the direction of a member of the orthogonal ennuple.

We may apply the notion of these rates of change in the scalar product of vectors in another way and obtain a geometrical deduction.

Put

$$T^l_{pq} = \tfrac{1}{2} g^{lt}\{(g_{pq})_t - (g_{pt})_q - (g_{qt})_p\}, \quad T_{spq} = g_{sl} T^l_{pq}. \tag{3.2}$$

Then (2.1′) gives

$$\nabla^l_{pq} = \begin{Bmatrix} l \\ pq \end{Bmatrix} + T^l_{pq}. \tag{3.3}$$

Now, the equations of the autoparallel described with respect to (2.1) are

$$\frac{d\lambda^t}{ds} + \nabla^t_{ij}\lambda^i\lambda^j = 0, \quad \lambda^l = \frac{dx^l}{ds}. \tag{3.4}$$

From (3.3) it follows that the autoparallel is a geodesic if

$$p^t \equiv T^t_{ij}\lambda^i\lambda^j = 0$$

Obviously, the geodesic curvature (Levi-Civita, 1927) of the autoparallel is the vector $-p^t$.

Put

$$p_T = \sum_T {}^T h_t p^t = T_{lij}\lambda^i\lambda^j {}_T h^l \tag{3.5}$$

$$\therefore \quad p_T = \{\tfrac{1}{2}(g_{ij})_l + [g_{ij}]_l\}\lambda^i\lambda^j {}_T h^l = \tfrac{1}{2} g_{ij/l}\lambda^i\lambda^j {}_T h^l.$$

Thus the condition that the autoparallel (3.4) is a geodesic is that the rate of change in the length of the unit vector λ, as the vector is given the parallel transport (2.1) in the direction of an arbitrary member of the orthogonal ennuple, is equal to twice the negative of the rate of change in the length of the vector when given the parallel transport (2.3) in the same direction, or that the length of the unit vector λ remains invariant when it is given the parallel transport (2.6) in an arbitrary direction of the ennuple.

In particular, it follows from (2.8) and (3.5) that the autoparallel described by a member ${}_{S}h$ of the ennuple is a geodesic if

$$\beta_{SST}-\beta_{STS}-\beta_{TSS}=0, \quad T=1,\ldots,n. \tag{3.6}$$

4. Let K^t_{jkl} and R^t_{jkl} be the curvature tensors formed with respect to Levi-Civita parallelism and the parallelism (2.1) respectively and put

$$K_{ijkl}=g_{it}K^t_{jkl}, \quad R_{ijkl}=g_{it}R^t_{jkl}.$$

As is well known, the Riemann-Christoffel tensor K_{ijkl} possesses the following properties:

$$K_{ijkl}+K_{ijlk}=0, \quad K_{ijkl}+K_{jikl}=0, \quad K_{ijkl}=K_{klij}, \quad K_{ijkl}+K_{iklj}+K_{iljk}=0. \tag{4.1}$$

From (3.3) we have

$$K^t_{jkl}-R^t_{jkl}=(T^t_{jk})_l-(T^t_{jl})_k+T^t_{sk}T^s_{jl}-T^t_{sl}T^s_{jk}. \tag{4.2}$$

Put

$$A_{ijkl}=(T_{ijk})_l-(T_{ijl})_k, \tag{4.3}$$

$$P_{ijkl}=g^{st}\{T_{sjk}T_{til}-T_{sjl}T_{tik}\}. \tag{4.4}$$

Substituting from (3.2) in (4.2), making use of the identity

$$(g^{st})_r=-g^{sp}g^{tq}(g_{pq})_r.$$

and multiplying by g_{it}, we obtain from (4.2)

$$K_{ijkl}-R_{ijkl}=A_{ijkl}+P_{ijkl}. \tag{4.5}$$

The tensor T_{rpq} is symmetric in p, q, and it is seen that P_{ijkl} possesses all the properties (4.1) while A_{ijkl} possesses only the first and the last of these properties. In view of the very nature of construction of the curvature tensor, a geometrical significance attaches to the tensor $A_{ijkl}+P_{ijkl}$. This significance, which is a special case of a more general one given earlier (Sen, 1948), is that this tensor arises in connection with the change in the scalar product of two vectors when the vectors are taken round a closed circuit, one of them by Levi-Civita parallelism and the other by the parallelism (2.2).

Evidently this tensor has come out in combination with Levi-Civita parallelism and the parallelism (2.1). We give below a method showing how it can be obtained independently of the Levi-Civita parallelism. However there is, in what follows, no speciality about (2.1) and the method is applicable to all parallelism with symmetric connection satisfying (3.3).

Let ξ^{ij} be a skew-symmetric contravariant tensor. If, as before, covariant derivative with respect to Levi-Civita parallelism is denoted by a crooked bracket followed by indices, then it follows from (3.3) that

$$(\xi^{ij})_j=\{\xi^{ij}\}_j+\xi^{kj}T^i_{kj}+\xi^{ik}T^j_{kj}$$

$$\therefore \quad (\xi^{ij})_{ji}=\{\{\xi^{ij}\}_j+\xi^{kj}T^i_{kj}+\xi^{ik}T^j_{kj}\}_i+[\{\xi^{kj}\}_j+\xi^{sj}T^k_{sj}+\xi^{ks}T^j_{sj}]T^i_{ki}.$$

Now, as ξ^{kj} is skew-symmetric and T^i_{kj} is symmetric in k, j,

$$\xi^{kj}T^i_{kj} = \xi^{ij}T^k_{ij} = 0,$$

$$\therefore \quad (\xi^{ij})_{ji} = \{\xi^{ij}\}_{ji} + \{\xi^{ik}\}_i T^j_{kj} + \xi^{ik}\{T^j_{kj}\}_i + \{\xi^{ki}\}_i T^j_{kj} + \xi^{ki}T^j_{ij}\, T^s_{ks}$$

In order to simplify this we notice the following

$$\{\xi^{ij}\}_{ji} = 0 \quad \text{(Levi-Civita, 1929)}$$

$$[\{\xi^{ik}\}_i + \{\xi^{ki}\}_i] = 0, \quad \xi^{ki}T^j_{ij}T^s_{ks} = 0$$

And as

$$\{T^j_{kj}\}_i = (T^j_{jk})_i + T^j_{tj}T^t_{ki},$$

$$\xi^{ik}\{T^j_{kj}\}_i = \xi^{ik}(T^j_{kj})_i$$

$$\therefore \quad (\xi^{ij})_{ji} = \xi^{lk}(T^j_{kj})_l = \xi^{lk}[(T^j_{kj})_l + T^j_{sk}\, T^s_{jl}] = -\xi^{lk}[(T^j_{lj})_k + T^j_{sl}\, T^s_{jk}].$$

Hence, by (4 2) and (4.5),

$$(\xi^{ij})_{ji} = \tfrac{1}{2}\xi^{lk}(K^j_{jkl} - R^j_{jkl}) = \tfrac{1}{2}\xi^{lk}g^{ij}(A_{ijkl} + P_{ijkl}).$$

And as P_{ijkl} is skew in i, j,

$$(\xi^{ij})_{ji} = \tfrac{1}{2}\xi^{lk}g^{ij}A_{ijkl} \tag{4.6}$$

Thus, the tensor A_{ijkl} arises in connection with the divergence, with respect to the parallelism (2.1), of a contravariant skew-symmetric tensor of rank two.

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NOTE ON A SIMPLE CASE OF FORCED TORSIONAL OSCILLATION OF A CIRCULAR-CYLINDER

By
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If we take the axis of z along the axis of the cylinder and use cylindrical coordinates (r, θ, z), we know that for torsional vibrations the displacement $v(=u_\theta)$ satisfies the equation (Love, 1920, p. 291)

$$\frac{\rho}{\mu}\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2}. \tag{1}$$

The other two equations are identically satisfied, it being assumed that the components of displacement u_r and u_z are zero and $v(=u_\theta)$ is independent of θ.

We find also that

$$\left.\begin{aligned} &\widehat{rr} = 0, \quad \widehat{\theta\theta} = 0, \quad \widehat{zz} = 0, \quad rz = 0, \\ &\widehat{\theta z} = \mu\frac{\partial v}{\partial z}, \quad \widehat{r\theta} = \mu\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right). \end{aligned}\right\} \tag{2}$$

The general solution of equation (1) involves Bessel's functions but there is a simple solution for *free* vibration in which Bessel's functions are not required (Love, 1920, p. 292) In this note it will be shown that there is a case of *forced* torsional vibration which can be solved without the use of Bessel's functions.

The problem can be stated as follows: The end $z = 0$ of the cylinder is fixed while the other end $z = l$ is acted on by a periodic shearing force proportional to $r \sin \Omega t$, the curved surface being free from stress at all times.

Here we are to obtain a solution of equation (1) subject to the conditions

$$v = 0, \quad z = 0, \quad t \geq 0, \tag{3.1}$$

$$\mu\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right)_{r=a} = 0, \quad t \geq 0, \tag{3.2}$$

$$\mu\left(\frac{\partial v}{\partial z}\right)_{z=l} = Qr \sin \Omega t, \quad t \geq 0, \quad (Q \text{ being a constant}) \tag{3.3}$$

and

$$\frac{\partial v}{\partial t} = 0, \quad t = 0. \tag{3.4}$$

We suppose,

$$v = v_1 + v_2$$

where v_1 is a particular solution of the equation (1) satisfying the conditions (3.1), (3.2) and (3.3) while v_2 is the function which satisfies equation (1) and the conditions (3.1) and (3.2) also makes $\mu(\partial v_2/\partial z)_{z=l} = 0$.

Let

$$v_1 = Qr(A \cos pz + B \sin pz) \sin \Omega t.$$

Now, v_1 satisfies equation (1) and the conditions (3.1), (3.2) and (3.3) if

$$A = 0 \quad p = \Omega/c \text{ where } c^2 = \mu/\rho$$

and

$$B = \frac{1}{\mu p \cos pl}.$$

We assume

$$v_2 = r\sum_{n=0}^{\infty}(A_n \cos q_n z + B_n \sin q_n z)(C_n \cos \alpha_n t + D_n \sin \alpha_n t).$$

v_2 satisfies equations (1), (3.1), (3.2) and makes

$$\mu\left(\frac{\partial v_2}{\partial z}\right)_{z=l} = 0,$$

if

$$A_n = 0, \quad q_n = \frac{2n+1}{2}\cdot\frac{\pi}{l} = \frac{\alpha_n}{c}.$$

Then from the condition (3.4), *viz.*,

$$\left(\frac{\partial v}{\partial t}\right)_{t=0} = \left(\frac{\partial v_1}{\partial t} + \frac{\partial v_2}{\partial t}\right)_{t=0} = 0$$

we get,

$$\sum_{n=0}^{\infty} B_n D_n \alpha_n \sin q_n z = -\, QB\Omega \sin pz$$

which gives

$$B_n D_n \alpha_n \int_0^l \sin^2 q_n z\,dz = -QB\Omega \int_0^l \sin pz \sin q_n z\,dz$$

or

$$B_n D_n = \frac{4(-1)^{n+1}\Omega Q}{(2n+1)\pi\mu c(q_n^2 - p^2)}.$$

Thus we get

$$v = Qr\left[\frac{c \sin(\Omega z/c)}{\mu\Omega \cos(\Omega l/c)} \sin \Omega t + \frac{4\Omega}{\pi\mu c}\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{(2n+1)(q_n^2 - p^2)} \sin q_n z(K_n \cos \alpha_n t + \sin \alpha_n t)\right].$$

K_n can be determined from the value of v at $t = 0$ for $0 < z \leq l$, and it is zero if $v = 0$ when $t = 0$.

The displacement due to the forced vibration has the tendency to increase indefinitely when $q_n \to p$ or $\alpha_n \to \Omega$.

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Reference

Love, A. E. H., (1920), *Mathematical Theory of Elasticity*

ON PAIRS OF FUNCTIONS WHICH ARE RECIPROCAL IN FOURIER-SINE TRANSFORMS

By

S. C. Mitra

(*Received January 31, 1949*)

1. Let $f(x)$ be of bounded variation in $(0, \infty)$ and tend to 0 as $x \to \infty$ Ramanujan (1915) has proved that

$$\sqrt{\beta}.[F_c(\beta)-F_c(3\beta)-F_c(5\beta)+F_c(7\beta)+\cdots] = \sqrt{\alpha}.[f(\alpha)-f(3\alpha)-f(5\alpha)+\cdots], \ (\alpha\beta = \tfrac{1}{2}\pi).$$

The theorem also remains true if $f(x)$ is continuous. We have

$$F_c(x) = \sqrt{\left(\frac{2}{\pi}\right)}\cdot\int_0^\infty f(t)\cos xt.dt$$

Let $f(t) = J_0(tx)$ and $\beta = \frac{1}{4}\pi$. We get

$$F_c(\pi/4) = \sqrt{\left(\frac{2}{\pi}\right)}\cdot\int^\infty J_0(tx)\cos\tfrac{1}{4}\pi t.dt = \sqrt{\left(\frac{2}{\pi}\right)}\cdot\frac{1}{\sqrt{(x^2-\frac{1}{16}\pi^2)}}, \ (x > \tfrac{1}{4}\pi)$$
$$= 0, \quad (x < \tfrac{1}{4}\pi).$$

Hence

$$F(t) = \{J_0(t.\tfrac{1}{2}\sqrt{\pi})-J_0(t.\tfrac{3}{2}\sqrt{\pi})-J_0(t.\tfrac{5}{2}\sqrt{\pi})+\cdots\}$$
$$= \sqrt{\left(\frac{2}{\pi}\right)}\left\{\frac{1}{\sqrt{(t^2-\frac{1}{4}\pi)}}-\frac{1}{\sqrt{(t^2-\frac{9}{4}\pi)}}-\frac{1}{\sqrt{(t^2-\frac{25}{4}\pi)}}+\cdots\right\} \tag{A}$$

We know that (Mitra, 1949) $F'(t)$ is self reciprocal in the Sine Transform. We have also (Titchmarsh, 1937, p. 203)

$$\int_0^\infty \frac{xJ_0(xt)}{1+x^2}\,dx = K_0(t); \quad \int_0^{\frac{1}{2}\pi} e^{-x\cos\varphi}d\varphi = \tfrac{1}{2}\pi\{I_0(x)-L_0(x)\}$$

and

$$\int_0^\infty \frac{J_0(xt)dx}{1+x^2} = \tfrac{1}{2}\pi\{I_0(t)-L_0(t)\}.$$

Let

$$\psi(t) = K_0(\tfrac{1}{2}t\sqrt{\pi})-K_0(\tfrac{3}{2}t\sqrt{\pi})-K_0(\tfrac{5}{2}t\sqrt{\pi})+K_0(\tfrac{7}{2}t\sqrt{\pi})+\cdots, \tag{B}$$

and

$$\chi(t) = \tfrac{1}{2}\pi[\{I_0(\tfrac{1}{2}t\sqrt{\pi})-L_0(\tfrac{1}{2}t\sqrt{\pi})\}-\{I_0(\tfrac{3}{2}t\sqrt{\pi})-L_0(\tfrac{3}{2}t\sqrt{\pi})\}-\cdots \tag{C}$$

Then it can be easily proved that

(1) The series (A), (B), (C) are uniformly convergent and (2) $I_0(x)-L_0(x) \sim 1/x$.

From what we have stated before, it follows that

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§ This has been proved in a previous paper which is to come out in the Bull Cal Math Soc

$$\int_0^\infty \frac{tF(xt)dt}{1+t^2} = \psi(x).$$

Multiplying both sides by sin kx and integrating with respect to x between the limits zero and infinity and changing the order of integration, we get

$$\sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty \frac{t}{1+t^2} dt \int_0^\infty F(xt) \sin kx.dx = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty \psi(x) \sin kx.dx$$

which reduces to

$$\int_0^\infty \frac{F(k/t)}{1+t^2} dt = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty \psi(x) \sin kx.dk.$$

Left hand side is

$$\int_0^\infty \frac{F(kt)}{1+t^2} dt = \tfrac{1}{2}\pi\{[I_0(\tfrac{1}{2}k\sqrt{\pi}) - L_0(\tfrac{1}{2}k\sqrt{\pi})] - [I_0(\tfrac{3}{2}k\sqrt{\pi}) - L_0(\tfrac{3}{2}k\sqrt{\pi})] - \cdots\} = \chi(k).$$

Thus we have,

$$\sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty \psi(x) \sin kx.dx = \chi(k).$$

In a similar manner, we can prove that

$$\begin{aligned} \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty \chi(x) \sin kx.dx &= \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty \frac{dt}{1+t^2} \int_0^\infty F(xt) \sin kx.dx \\ &= \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty \frac{dt}{t(1+t^2)} \int_0^\infty F(x) \sin \frac{k}{t} x.dx \\ &= \int_0^\infty \frac{F(k/t)dt}{t(1+t^2)} = \int \frac{tF(kt)dt}{1+t^2} \\ &= [K_0(\tfrac{1}{2}k\sqrt{\pi}) - K_0(\tfrac{3}{2}k\sqrt{\pi}) - K_0(\tfrac{5}{2}k\sqrt{\pi}) + \cdots] = \psi(k). \end{aligned}$$

2. We know that (Titchmarsh, 1937, 202)

$$\int_0^\infty \frac{tJ_0(xt)}{(1+t^4)^{\frac{1}{2}}} dt = J_0(x/\sqrt{2})K_0(x/\sqrt{2})$$

Also it can be easily proved that

$$\int_0^\infty \frac{J_0(xt)dt}{(1+t^4)^{\frac{1}{2}}}$$

exists and let us denote the integral by $F_1(x/\sqrt{2})$. Then $F_1(x) \sim 1/\sqrt{x}$.

Let us put

$$\psi_1(x) = J_0\left(\frac{x\sqrt{\pi}}{2\sqrt{2}}\right) K_0\left(\frac{x\sqrt{\pi}}{2\sqrt{2}}\right) - J_0\left(\frac{3x\sqrt{\pi}}{2\sqrt{2}}\right) K_0\left(\frac{3x\sqrt{\pi}}{2\sqrt{2}}\right) - \cdots$$

and

$$\chi_1(x) = F_1\left(\frac{x\sqrt{\pi}}{2\sqrt{2}}\right) - F_1\left(\frac{3x\sqrt{\pi}}{2\sqrt{2}}\right) - F_1\left(\frac{5x\sqrt{\pi}}{2\sqrt{2}}\right) + \cdots \tag{D}$$

Then proceeding exactly as before, we have

$$\sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty \psi_1(x) \sin kx.dx = \chi_1(k)$$

and

$$\sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty \chi_1(x) \sin kx.dx = \psi_1(k).$$

3. Let us next consider the integral

$$\int_0^\infty \frac{t^\mu J_0(tx)dt}{(t^4+1)^{\frac{1}{2}\mu+\frac{1}{4}}} = F_2(x), \quad (\mu \geq -\tfrac{1}{2}).$$

The integral exists and is asymptotic to $1/\sqrt{x}$.

Putting

$$\psi_2(x) = F_2(\tfrac{1}{2}x\sqrt{\pi}) - F_2(\tfrac{3}{2}x\sqrt{\pi}) - F_3(\tfrac{5}{2}x\sqrt{\pi}) + \cdots \tag{E}$$

we get

$$\int_0^\infty \frac{t^\mu F(xt)dt}{(1+t^4)^{\frac{1}{2}\mu+\frac{1}{4}}} = \psi_2(x).$$

Therefore

$$\sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty \psi_2(x) \sin kx.dx = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty \frac{t^\mu dt}{(1+t^4)^{\frac{1}{2}\mu+\frac{1}{4}}} \int_0^\infty F(xt) \sin kxdx$$

$$= \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty \frac{t^{\mu-1}dt}{(1+t^4)^{\frac{1}{2}\mu+\frac{1}{4}}} \int_0^\infty F(x) \sin \frac{k}{t}x.dx$$

$$= \int_0^\infty \frac{t^{\mu-1}F(k/t)dt}{(1+t^4)^{\frac{1}{2}\mu+\frac{1}{4}}} = \int_0^\infty \frac{t^\mu F(kt)dt}{(1+t^4)^{\frac{1}{2}\mu+\frac{1}{4}}} = \psi_3(k);$$

showing that $\psi_2(x)$ is self-reciprocal in the Sine-transform.

It now remains to justify the change in the order of integration.

The following conditions are easily satisfied. Denoting the integrand by $F(x,t)$, we find that (1) the integrand is a continuous function of x and t, (2) the repeated integrals between finite limits always exist and are equal.

(3) $\int_a^\infty f(x, t)dx$ is uniformly convergent in an arbitrary interval;

(4) $\int_b^\infty f(x, t)dt$ satisfies the same condition;

(5) $\int_a^\infty \int_b^T f(x, t)dt$ coverges uniformly in an unlimited interval; or

(6) $\int_b^\infty \int_a^X f(x, t)dx$ converges uniformly in an unlimited interval.

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Mitra, S. C., (1949), *Bull. Cal Math Soc.*, **41**, 1

Ramanujan, S., (1915), *Messenger of Mathematics*, **44**, 10-18

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ON PERCENTAGE POINTS OF INCOMPLETE BETAFUNCTIONS AND χ^2 DISTRIBUTION

BY

D. P. BANERJEE

(*Received January 3, 1949*)

In his introduction to the Tables of the Incomplete Beta-functions Karl Pearson (1934) says that 'No single method has hitherto been discovered for evaluating numerically the incomplete Beta-function for all the values of p and q'. L. J. Comrie and Hartley (1941) tried to solve the problem indirectly. Here a new and direct method is given.

1 The 'normalised' incomplete Beta-function is

$$P(x,\,p,\,q) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)}\int_0^x x^{p-1}(1-x)^{q-1}dx,$$

from which

$$\frac{dP}{dx} = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)}\,x^{p-1}(1-x)^{q-1}.$$

Hence

$$\frac{dx}{dP} = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.\frac{1}{x^{p-1}(1-x)^{q-1}}.$$

When $x = \cdot 5$, let $P = \alpha$, and since

$$x = f(P-\alpha) = \sum_0^\infty f^r(0)\,\frac{(P-\alpha)^r}{r\,!}$$

then $f(0) = \cdot 5$.

Differentiating we have

$$\frac{d^2x}{dP^2} = \left(\frac{dx}{dP}\right)^2 \frac{[x(p+q-2)-(p-1)]}{x(1-x)}.$$

Now

$$f'(0) = \left(\frac{dx}{dP}\right)_{P=\alpha} = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)(\cdot 5)^{p+q-2}}.$$

Hence

$$f''(0) = \frac{\{\Gamma(p)\}^2\{\Gamma(q)\}^2[\cdot 5(p+q-2)-(p-1)]}{\{\Gamma(p+q)\}^2(\cdot 5)^{2p+2q-2}}.$$

In this way we can calculate $f^r(0)$ up to the value of r for which $f^r(0)(P-\alpha)^r/r\,!$ is small. Thus we get the value x for given values of p, q and P.

Example. Let $p = 3$, $q = 2$, to find x when $P = \cdot 32$, $\alpha = \cdot 31$, we have $f'(0) = \cdot 7$, $f''(0) = \cdot 9$. Hence $x = \cdot 5 + \cdot 007 + \cdot 00005 = \cdot 507$ approx.

2. The probability distribution of χ^2 having ν degrees of freedom may be written as

$$P = \frac{\{\Gamma(\frac{1}{2})\}^{\nu/2}}{\Gamma(\nu/2)} \int_x^\infty x^{\frac{1}{2}\nu-1} e^{-\frac{1}{2}x} dx.$$

When $x = 1$, let $P = \alpha$ and

$$x = f(P-\alpha) = \sum_0^\infty f^r(0) \frac{(P-\alpha)^r}{r!}.$$

Now

$$\frac{dP}{dx} = \frac{\{\Gamma(\frac{1}{2})\}^{\nu/2}}{\Gamma(\nu/2)} x^{\frac{1}{2}\nu-1} e^{-\frac{1}{2}x}$$

Hence

$$\frac{dx}{dP} = \frac{\Gamma(\nu/2)}{\{\Gamma(\frac{1}{2})\}^{\nu/2}} \frac{e^{\frac{1}{2}x}}{x^{\frac{1}{2}\nu-1}} \text{ and } \frac{d^2x}{dP^2} = \frac{x-\nu+2}{2x}\left(\frac{dx}{dP}\right)^2$$

Finding $f^r(0)$ for $r = 1, 2, 3, \ldots,$ we get the value of

$$x = 1 + \sum_1^\infty f^r(0) \frac{(P-\alpha)^r}{r!}$$

provided $P-\alpha < 1$.

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References

Comrie, L. J. and Hartley, H. O., (1941-42), *Biometrica*, 32, 151

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CALCUTTA MATHEMATICAL SOCIETY

Report of the Council for the year 1948 to the Annual General Meeting of the Society

The Council of the Calcutta Mathematical Society has the pleasure to submit the following report on the general concerns of the Society for the year 1948 as required by the provisions of Rule 25.

THE COUNCIL: The Council of the Society for the year 1948 consisting of the officers and members elected at the last Annual General meeting together with the Editorial Secretary, was constituted as follows:

President

Prof. A. C. Banerjee

Vice-Presidents

Prof. N. R. Sen
Dr. S. S. Pillai
Prof. N. M. Basu
Prof. B. B. Sen
Prof. V. V. Narlikar

Treasurer

Mr. S. C. Ghosh

Secretary

Mr. U. R. Burman

Editorial Secretary

Mr. P. K. Ghosh

Other Members of the Council

Prof. F. W. Levi
Prof. S. N. Bose
Dr. N. G. Shabde
Mr. A. C. Choudhury
Dr. B. R Seth
Dr. P. N. Das Gupta
Dr. B. S. Ray
Dr. R. N. Sen
Dr. S. Ghosh
Prof. M. R. Siddiqi
Prof. C. N. Srinivasiengar
Mr. B. N. Mukherjee

GENERAL: The general activities of the Society have been conducted in much the same form as in the last few years. The Council while realising the difficulties in restoring and maintaining the normal activities of the Society under the prevailing conditions earnestly hopes that not only these difficulties will be overcome before long but means will be found out for further development of the aims and objects of the Society.

MEMBERSHIP: The Council records with regret the loss of one of the members of the Society in the death of Prof. L K. Ray of the Islamia College, Calcutta.

The Council also reports with pleasure that during the year under review the following distinguished persons have been elected as Honorary members of the Society.

- Sir G. I. Taylor (England)
- Prof. E. A. Milne (England)
- Prof J. E. Littlewood (England)
- Prof. E. Schrödinger (Eire)
- Prof. E. Cartan (France)
- Prof. Hermann Weyl.(U.S.A.)
- Prof. I. Vinogradov (U.S S.R.)
- Prof. B. Lindblad (Sweden)
- Prof. de la Vallée Poussin (Belgium)

The Council desires to add that the Society cannot achieve its objects for which it exists without adequate numerical and financial strength. It is also worthwhile to make every effort to bring into the Society as many as possible of those persons who would be glad to benefit by its activities once having been made properly aware of them. The Council believes that the number of such persons is far in excess of the number who do at present become members.

MEETINGS DURING 1948: The Council held *five* meetings during the year and there were *six* Ordinary General meetings which were devoted to the reading of original papers communicated to the Society for publication in the Bulletin.

PUBLICATIONS: During the year 1948 the Society has published six numbers of the Bulletin Vol. 39, Nos. 2, 3, 4, and Vol. 40, Nos. 1, 2, 3. The Society received with grateful thanks a sum of Rs. 550/- from the Rockefeller Foundation Grant distributed through the National Institute of Sciences of India. The council reports with pleasure that this grant has largely contributed towards the improvement of the Bulletin.

The authorities of the Calcutta University have continued to lend their usual magnanimous services in printing the Bulletin free of charge, and the officers and members of the staff of the University Press have given their every sympathetic and active cooperation in bringing out the Bulletin, as many as six issues of it, during the year under review. The Council takes this opportunity to offer them its very sincere thanks.

EXCHANGE OF PUBLICATIONS: The transmission of the Society's publications to various countries of the world has been carried on in the usual manner, and some new exchange relations have also been established during the year under review. In opening up new exchange relations the object of the Council has always been to ensure that the Society's publications are available to all scientific workers throughout the world, who require to consult them, as far as possible. The Council desires to report here that a certain number of German Journals, specially the Astrophysical ones have resumed publication during the year, and every endeavour is being made to have them on an exchange basis.

FINANCE: The annual accounts of the Society for the year 1948 have been presented to the Council in the standardized form by the auditors, Dr. S. K. Basu and Mr. B. C. Chatterjee. The Council gratefully acknowledges its indebtedness to them for their honorary services. A comparison of the receipt and disbursement accounts would show a deficit for the year. The deficit is caused partly by increases in salaries and mainly by the heavy expenditure on account of papers, blocks and types, and also on account of postage due to the increased circulation and despatches of back issues of the Bulletin to fill up some wartime gaps. There will also be noticed a decline on the receipt side under the head "membership subscriptions" and the council has every reason to hope that not only the arrear subscriptions will be fully realised during the course of the next year but also that there will be a substantial increase in the number of members.

CALCUTTA MATHEMATICAL SOCIETY

RECEIPTS AND DISBURSEMENTS ACCOUNTS OF THE CALCUTTA MATHEMATICAL SOCIETY FOR THE YEAR ENDING 31ST DECEMBER, 1948.

Receipts

			Rs. As. P.
1. Opening balance			
(a) With Secretary			
(i) In cash	17 5 0		
(ii) In stamps	2 0 9		
		19 5 9	
(b) Balance at Bank			
(i) Imperial Bank of India	1299 7 6		
(ii) Do. (K.K G. P. Fund)	317 11 8		
(iii) Bengal Central Bank	951 5 6		
(iv) P. O. Savings Bank	587 0 0		
		3155 8 8	
(c) G P Notes (General Fund) (Face value Rs. 6,000) ...		5663 11 6	
(d) G. P Notes (K.K G.P Fund) (Face value Rs. 2,000) .		1937 7 9	
			10,776 1 8
2. Membership Subscriptions ...		687 2 0	
3. Admission Fees		10 0 0	
4. Sale Proceeds			
(a) Actually realised .		953 12 9	
(b) In Suspense A/C ($ 27 00) ...		88 12 0	
5 Rockefeller Foundation Grant		550 0 0	
			2289 10 9
6 Interest			
(a) G. P Notes (General Fund) ...		360 0 0	
(b) Do (K K G. P. Fund) ..		80 10 0	
(c) P. O Savings Bank ...		8 13 0	
			449 7 0
TOTAL Rs.			13,515 3 5

Disbursements

			Rs. As. P
1 Establishment ...		952 8 0	
2 Meetings ...		93 14 0	
3. Books and Journals (including binding charges)		302 1 4	
4. Bulletins			
(a) Papers, Blocks and Types	946 14 3		
(b) Postage	215 0 0		
		1161 14 3	
5 Printing and Stationery ...		30 8 0	
6 Postage	63 1 6		
Less stamps in hand	0 14 6		
		62 3 0	
7. Bank Charges ...		19 0 0	
8 Miscellaneous (including conveyance charges)...		50 0 9	
			2,672 1 4
Closing Balance			
(a) With Secretary			
(i) In cash	12 4 0		
(ii) In stamps	0 14 6		
		13 2 6	
(b) Balance at Bank			
(i) Imperial Bank of India	1918 3 6		
(ii) Do. (K K. G. P. Fund)	397 9 8		
(iii) Bengal Central Bank	228 6 2		
In suspense ($ 27 00)	88 12 0		
(iv) P O Savings Bank	595 13 0		
		3,228 12 4	
(c) G. P. Notes (General Fund) (Face value Rs 6,000) ...		5663 11 6	
(d) G P Notes (K K. G P. Fund) (Face value Rs 2,000) ..		1937 7 9	
			10,843 2 1
TOTAL ... Rs.			13,515 3 5

To

The Members of the Calcutta Mathematical Society.

We have examined the above Balance Sheet with the Books and Vouchers relating thereto and certify it to be correctly drawn up therefrom and in accordance with the information and explanations given to us.

S. K Basu
B. C. Chatterjee *Auditors*

A STUDY OF THE GENERALISED LAPLACE INTEGRAL—II

By

S. K. Bose

(Communicated by the Secretary—Received July 21, 1948)

1.1 We* now establish another theorem in which the variable in the intermediary function $h(x)$ is raised to some power. This theorem also gives rise to a new function, special cases of which are some of the well known functions occuring in Analysis. The theorem itself is quite general and includes the results of Shastri and Humbert in the theory of Operational Calculus.

1.2. Theorem II. If

$$\varphi(p) \overset{k}{\underset{m}{\doteqdot}} h(x)$$

and

$$p^{1-\lambda+\mu}h(p^\mu) \doteqdot g(x),$$

then, provided the integral is convergent,

$$\frac{1}{p}\varphi(p) = \int_0^\infty W_\lambda^\mu(p/s^\mu)g(s)ds, \quad \text{for} \quad \mu \geqq 1, \tag{1}$$

where

$$W_\lambda^\mu(p/s^\mu) = \frac{1}{(2p)^{\lambda/\mu}}\sum_{r=0}^{\infty}\frac{(-)^r\,\Gamma(\lambda/\mu+m+r/\mu+\frac{1}{4})\Gamma(\lambda/\mu-m+r/\mu+\frac{1}{4})}{r!\;\Gamma(\lambda/\mu-k+r/\mu+\frac{3}{4})}$$

$$\times\,(s^\mu/2p)^{r/\mu}\,.\,{}_2F_1\left\{\begin{matrix}\lambda/\mu+m+r/\mu+\frac{1}{4},\ \lambda/\mu-m+r/\mu+\frac{1}{4}\\ \lambda/\mu-k+r/\mu+\frac{3}{4}\end{matrix};\ \frac{1}{2}\right\},\quad \mathbf{R}(\lambda/\mu\pm m+\tfrac{1}{4})>0.$$

In case, however, if $\mu = 1$, we have (B, Eqn 1 2)

$$W_\lambda^1(p/s) = \frac{1}{(2p)^\lambda}\frac{\Gamma(\lambda+m+\frac{1}{4})\,\Gamma(\lambda-m+\frac{1}{4})}{\Gamma(\lambda-k+\frac{3}{4})}\ {}_2F_1\left\{\begin{matrix}\lambda+m+\frac{1}{4},\lambda-m+\frac{1}{4}\\ \lambda-k+\frac{3}{4}\end{matrix};\ \frac{1}{2}-\frac{s}{2p}\right\}$$

1.3. *Proof of Theorem II.*

We have

$$\frac{1}{p}\varphi(p) = \int_0^\infty (2xp)^{-\frac{1}{4}}W_{k,m}(2px)h(x)dx \tag{1}$$

and

$$x^{1-\lambda+\mu}h(x^\mu) = x\int_0^\infty e^{-xs}g(s)ds. \tag{2}$$

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[*] This paper is in continuation to my previous paper (Bose, 1949) Henceforth this paper will be denoted by B.

Substituting the values of $h(x)$ from (2) in (1), we get

$$\frac{1}{p}\varphi(p) = \int_0^\infty (2xp)^{-\frac{1}{4}} W_{k,m}(2px)x^{\lambda/\mu-1}dx \int_0^\infty \exp(-x^{1/\mu}s).g(s)ds.$$

On changing the order of integration, we have

$$\frac{1}{p}\varphi(p) = \frac{1}{(2p)^{\lambda/\mu}} \int_0^\infty g(s)\left[\int_0^\infty \exp\{-(x/2p)^{1/\mu}s\}\, x^{\lambda/\mu-\frac{1}{4}-1}\, W_{k,m}(x)dx\right]ds. \tag{3}$$

Now, expanding the exponential function and integrating them term by term, which is justifiable, we have (1.2.1).

To justify the change in the order of integration in (3), suppose that

$$\theta(s) = g(s)\int_0^\infty \exp(-x^{1/\mu}s)\,(2px)^{-\frac{1}{4}}x^{\lambda/\mu-1}W_{k,m}(2px)dx$$

and

$$\psi(x) = (2xp)^{-\frac{1}{4}}W_{k,m}(2px)x^{\lambda/\mu-1}\int_0^A \exp(-x^{1/\mu}s).g(s)ds$$

where A is small. But we have

$$\theta(s) = g(s)W_\lambda^\mu(p/s^2)$$

Now, I have shown below that

$$W_\lambda^\mu(p/s^\mu) = O(1) \text{ for small } s.$$

It is evident that $\theta(s)$ is uniformly convergent in $s \geq 0$, if $\mathbf{R}(\mu_1) > 0$, where $g(s) = O(s^{\mu_1})$ for small s and $\mathbf{R}(\lambda/\mu \pm m + \frac{1}{4}) > 0$.

$\psi(x)$ also converges uniformly in $x \geq 0$, provided that $\mathbf{R}(\mu_1) > -1$ and $\mathbf{R}(\lambda/\mu \pm m - \frac{3}{4}) > 0$.

Again, consider the integral

$$I = \int_T^\infty \left|(2xp)^{-\frac{1}{4}}W_{k,m}(2px)x^{\lambda/\mu-1}\right|dx.\int_{T'}^\infty \left|\exp(-x^{1/\mu}s)\; g(s)\right|ds$$

where T and T' are large.

For large values of s, I tends to zero when $\mathbf{R}(p) > 0$, and when

$$\int_0^\infty \exp(-x^{1/\mu}s)\,|\,g(s)\,|\,ds$$

is convergent.

Hence the change in the order of integration is justified under the conditions,

$$\mathbf{R}(\lambda/\mu \pm m + \tfrac{1}{4}) > 0, \quad \mathbf{R}(p) > 0, \quad \mathbf{R}(\mu_1) > 0, \quad \int_0^\infty \exp(-x^{1/\mu}s)\,|\,g(s)\;ds$$

is assumed to be convergent.

1.4. Corollary.

If in (1.2.1), we put $k = \frac{1}{4}$, $m = \pm\frac{1}{4}$ Whittaker transform reduces to Laplace transform and our theorem reduces to:—

If
$$\varphi(p) \doteqdot h(x)$$
and
$$p^{1-\lambda+\mu} h(p^{\mu}) \doteqdot g(x).$$
then, provided the integral is convergent

$$\frac{1}{p}\varphi(p) = \int_0^\infty W_\lambda^\mu(p/s^\mu) g(s)\,ds, \quad \text{for} \quad \mu \geq 1,\ \mathbf{R}(\lambda) > 0,$$

where
$$W_\lambda^\mu(p/s^\mu) = \frac{1}{p^{\lambda/\mu}} \sum_{r=0}^{\infty} \frac{(-)^r \Gamma\{(\lambda+r)/\mu\}}{r!} (s/p^{1/\mu})^r.$$

1.5. *Properties of $W_\lambda^\mu(x)$.*

The function $W_\lambda^\mu(x)$ is pretty general and includes, as special cases, some of the important functions in Analysis. In the first place we note that we have a meaning to this function only when $\mu \geq 1$ It is easy to see that for $k = \frac{1}{4}$, $m = \pm\frac{1}{4}$, and $\mu = 2$, we get
$$W_\lambda^2(x) = 2^{1-\frac{1}{2}\lambda} p^{-\frac{1}{2}\lambda} \Gamma(\lambda) e^{1/8x} D_{-\lambda}\{1/(2x)^{\frac{1}{2}}\},$$
where $D_n(x)$ is Weber's Parabolic Cylinder Function.

Another case of reduction to a finite series of parabolic cylinder function occurs if we consider the theorem in K-transform.

The behaviour of $W_\lambda^\mu(x)$, for small and large values of x, can easily be investigated. We have, in fact,
$$W_\lambda^\mu(p/x^\mu) = \frac{1}{(2p)^{\lambda/\mu}} \int_0^\infty \exp\{-xs^{1/\mu}/(2p)^{1/\mu}\}.s^{\lambda/\mu-5/4} W_{k,m}(s)\,ds.$$
It follows from our definition of $W_\lambda^\mu(p/x^\mu)$, that
$$W_\lambda^\mu(p/x^\mu) = O(1) \text{ for small } x.$$
If we make the transformation
$$s/2p = y^\mu,$$
we get
$$W_\lambda^\mu(p/x^\mu) = \frac{1}{(2p)^{\lambda/\mu}} \int_0^\infty e^{-xy}(2py^\mu)^{\lambda/\mu-5/4} W_{k,m}(2py^\mu) 2\mu p y^{\mu-1}\,dy$$
$$= \frac{\mu}{(2p)^{\frac{1}{4}}} \int_0^\infty e^{-xy} y^{\lambda-\frac{1}{4}\mu-1} W_{k,m}(2py^\mu)\,dy.$$
The function
$$y^{\lambda-\frac{1}{4}\mu-1} W_{k,m}(2py^\mu)$$
satisfies all the conditions of Watson's lemma (Watson, 1921) regarding the asymptotic expansion of function representable by Laplace's integral and is $O(y^{\lambda\pm m\mu+\frac{1}{4}\mu-1})$ for small y.

If $(\lambda \pm m\mu + \frac{1}{4}\mu)$ is not an integer, we have from Watson's lemma
$$W_\lambda^\mu(p/x^\mu) = O(x^{-\lambda\pm m\mu-\frac{1}{4}\mu}) \text{ for large } x.$$

(*a*) When $\mu = 2$, $\lambda = -m_1$, and $m = \pm\frac{1}{4}$, $k = \frac{1}{4}$ in (1.2 1) and further noting that

$$\sum_{r=0}^{\infty}\frac{(-)^r}{r!}\Gamma(\tfrac12 r+\tfrac12\lambda).(1/x)^{\frac12 r+\frac12\lambda}=2\Gamma(\lambda)(2x)^{-\frac12\lambda}e^{1/8x}D_{-\lambda}\{1/(2x)^{\frac12}\}$$

we get, the theorem due to Shastri (1945).

(*b*) When $\mu=1$, $\lambda=m_1+2$ and $k=\frac14$, $m=\pm\frac14$ in (1.2.1), we get another theorem of Shastri (1944).

(*c*) When $\mu=1$, $\lambda=2$ and $k=\frac14$, $m=\pm\frac14$, we get result due to Humbert (1937).

1.7. *Evamples*

(*i*) If

$$g(s)=\frac{2^{\nu}s^{2\nu-2\lambda}}{\Gamma(2\nu-2\lambda+1)}e^{-\frac12 s^2}.{}_1F_1[-2\lambda+\tfrac12,\ \nu-\lambda+1\,;\ \tfrac12 s^2]$$

then for $\mu=2$,

$$x^{-\lambda+3}h(x^2)=W_{-\nu,\lambda}(x^2/2)e^{\frac12 x^2}x^{2\lambda},\ \mathbf{R}(\nu\pm\lambda+\tfrac12)>0,$$

for

$$\frac{2^{\nu}s^{2\nu-2\lambda}.e^{-\frac12 s^2}}{\Gamma(2\nu-2\lambda+1)}.{}_1F_1\left\{\begin{matrix}-2\lambda+\frac12\\ \nu-\lambda+1\end{matrix};\ \tfrac12 s^2\right\}\doteqdot W_{-\nu\,\lambda}(x^2/2)e^{\frac12 x^2}x^{2\lambda}.$$

Therefore (B, Eqn. 1.2.8)

$$\frac1p\varphi(p)=\frac{1}{\Gamma(\frac12+\nu+\lambda)\Gamma(\frac12+\nu-\lambda)}\Bigg[\frac{\Gamma(-2\lambda)\Gamma(2\lambda+1)}{2^{\lambda+\frac14}}(2p)^{-5\lambda/2}$$

$$\times\sum_{r=0}^{\infty}\frac{\Gamma(r+\nu+\frac12+\lambda)\Gamma(5\lambda/2+m+r+\frac14)\Gamma(5\lambda/2-m+r+\frac14)}{r!\ \Gamma(r+2\lambda+1)\Gamma(5\lambda/2-k+r+\frac34)(4p)^r}$$

$$\times{}_2F_1\left\{\begin{matrix}5\lambda/2+m+r+\frac14,\ 5\lambda/2-m+r+\frac14\\ 5\lambda/2-k+r+\frac34\end{matrix};\ \tfrac12\right\}+\frac{\Gamma(2\lambda)\Gamma(-2\lambda+1)(2p)^{-\frac12\lambda}}{2^{-\lambda+\frac14}}$$

$$\times\sum_{r=0}^{\infty}\frac{\Gamma(r-\lambda+\frac12+\nu)\Gamma(\frac12\lambda+m+r+\frac14)\Gamma(\frac12\lambda-m+r+\frac14)}{r!\,\Gamma(r-2\lambda+1)\Gamma(\frac12\lambda-k+r+\frac34)(4p)^r}$$

$$\times{}_2F_1\left\{\begin{matrix}\frac12\lambda+m+r+\frac14,\ \frac12\lambda-m+r+\frac14\\ \frac12\lambda-k+r+\frac34\end{matrix};\ \tfrac12\right\}\Bigg],\ \mathbf{R}(3\lambda/2\pm m\pm\lambda+\tfrac14)>0,\ \mathbf{R}(p)>0\text{ and }|p|>\tfrac12.$$

Hence applying Theorem II, with $\mu=2$, we get

$$\frac{2^{\nu}}{\Gamma(2\nu-2\lambda+1)}\int_0^{\infty}s^{2\nu-2\lambda}e^{-\frac12 s^2}.{}_1F_1\left\{\begin{matrix}-2\lambda+\frac12\\ \nu-\lambda+1\end{matrix};\ \tfrac12 s^2\right\}W_{\lambda}^2((p/s^2)ds$$

$$=\frac{1}{\Gamma(\frac12+\nu+\lambda)\,\Gamma(\frac12+\nu-\lambda)}\Bigg[\frac{\Gamma(-2\lambda)\Gamma(2\lambda+1)}{2^{\lambda+\frac14}}(2p)^{-5\lambda/2}\sum_{r=0}^{\infty}\frac{\Gamma(r+\nu+\frac12+\lambda)}{r!\ \Gamma(r+2\lambda+1)}$$

$$\times\frac{\Gamma(5\lambda/2+m+r+\frac14)\Gamma(5\lambda/2-m+r+\frac14)}{\Gamma(5\lambda/2-k+r+\frac34)(4p)^r}.{}_2F_1\left\{\begin{matrix}5\lambda/2+m+r+\frac14,\ 5\lambda/2-m+r+\frac14\\ 5\lambda/2-k+r+\frac34\end{matrix};\ \tfrac12\right\}$$

$$+\frac{\Gamma(2\lambda)\Gamma(-2\lambda+1)(2p)^{-\frac12\lambda}}{2^{-\lambda+\frac14}}\sum_{r=0}^{\infty}\frac{\Gamma(r-\lambda+\frac12+\nu)\Gamma(\frac12\lambda+m+r+\frac14)\Gamma(\frac12\lambda-m+r+\frac14)}{r!\ \Gamma(r-2\lambda+1)\Gamma(\frac12\lambda-k+r+\frac34)(4p)^r}$$

$$\times{}_2F_1\left\{\begin{matrix}\frac12\lambda+m+r+\frac14,\ \frac12\lambda-m+r+\frac14\\ \frac12\lambda-k+r+\frac34\end{matrix},\ \right\}\Bigg].$$

provided (2λ) and $(\lambda \pm 2m + \frac12)$ are not integers, $\mathbf{R}(\frac12\lambda \pm m + \frac14) > 0$,, $\mathbf{R}(5\lambda/2 \pm m + \frac14) > 0$, $\mathbf{R}(\nu - \lambda + \frac12) > 0$, $\mathbf{R}(p) > 0$ and $|p| > \frac12$.

(*ii*) If

$$g(s) = \frac{\sqrt{\pi}\, s^n J_{n-1}(s)}{2^n \Gamma(n+\frac12)}$$

then with $\mu = 2$, we have

$$x^{3-\lambda} h(x^2) = \frac{x^2}{(1+x^2)^{n+\frac12}}, \quad \mathbf{R}(n) > 0$$

Therefore (B, Eqn, 1 2.20)

$$\frac{1}{p}\varphi(p) = \frac{\Gamma(2m+1)\Gamma(-2m)}{\Gamma(m+\frac12-k)\,\Gamma(-m+\frac12-k)} \sum_{r=0}^{\infty} \frac{\Gamma(r+m+\frac12-k)}{r!\;\Gamma(r+2m+1)} (2p)^{r+m+\frac12}$$

$$\times\, \Gamma(\tfrac12\lambda + m + r + \tfrac34) e^{\frac12 p}\, p^{-\frac12(\frac12\lambda+m-n+r+5/4)}\, W_{\begin{cases}-\frac12(\frac12\lambda+m+r+n+\frac14),\\ \;\frac12(\frac12\lambda+m-n+r+\frac14)\end{cases}}$$

$$\mathbf{R}(\pm m + \tfrac12\lambda + \tfrac34) > 0 \quad \text{and} \quad \mathbf{R}(p) > 0$$

Hence by the Theorem II with $\mu = 2$, we get

$$\frac{\sqrt{\pi}}{2^n \Gamma(n+\frac12)} \int_0^\infty s^n J_{n-1}(s) W_\lambda^2(p/s^2)\,ds$$

is equal to (1) provided $(2m)$ is not an integer, $\mathbf{R}(n - \lambda \pm 2m - 1) < 0$, $\mathbf{R}(\frac12\lambda \pm m + \frac14) > 0$ and $\mathbf{R}(n) > 0$.

On putting $k = \frac14$, $m = \pm\frac14$ and $\lambda = 2m$, we get Shastri's integral (1945).

(*iii*) If

$$g(s) = e^{-s^2/4} \sum_{t=0}^{n_1} \frac{(-)^{n_1-t}\, n_1!\; s^{n_1-t}}{t!\;[(n_1-t)!]^2} D_{n_1-t}(s)$$

then

$$h(x^2)x^{3-\lambda} = x^{n_1+1} e^{x^2/4} D_{-n_1-1}(x)^* \text{ with } \mu = 2,$$

Thus

$$\frac{1}{p}\varphi(p) = (-)^l l! \int_0^\infty (2xp)^{\frac12 n+\frac14} e^{-px} L_l^n(2px) h(x)\,dx$$

$$= 2(-)^l l!\; \Gamma(n+l+1) \sum_{r=0}^{\infty} \frac{(-)^r (2p)^{\frac12 n+r+\frac14}}{(l-r)!\; r!\; \Gamma(1+n+r)}$$

$$\times \int_0^\infty x^{\lambda+n+n_1+2r+\frac12-1} e^{(\frac14-p)x^2} D_{-n-1}(x)\,dx$$

On using the integral of Goldstein (1932)

$$\int_0^\infty y^{\nu-1} e^{(\frac14-\alpha)y^2} D_{\mu-1}(y)\,dy = \frac{\sqrt{\pi}.2^{\frac12\mu-\nu-\frac12}\,\Gamma(\nu)}{\gamma^{\frac12\nu}\,\Gamma(\frac12\nu-\frac12\mu+1)} .{}_2F_1\left\{\begin{matrix}\frac12\nu,\ \frac12(1-\mu)\\ \frac12(\nu-\mu)+1\end{matrix},\ 1-\frac{1}{2\alpha}\right\} \qquad \text{(A)}$$

$$\nu > 0,\ \mathbf{R}(\gamma) > 0 \text{ and } |\arg\alpha| < \tfrac12\pi,$$

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* See H., p. 53.

we have

$$\frac{1}{p}\varphi(p) = \frac{2\sqrt{\pi}.(-)^l l!\,\Gamma(n+l+1)}{2^{3n_1/2+\lambda+\frac{1}{2}n+3/2}\,p^{\frac{1}{2}(\lambda+n_1)}}\sum_{r=0}^{\infty}\frac{(-)^r 2^{-r}\Gamma(\lambda+n+n_1+2r+\frac{1}{2})}{(l-r)!\;r!\;\Gamma(1+n+r)\Gamma(\frac{1}{2}\lambda+\frac{1}{2}n+n_1+r+5/4)}$$

$$\times\,{}_2F_1\left\{\begin{matrix}\frac{1}{2}\lambda+\frac{1}{2}n+\frac{1}{2}n_1+r+\frac{1}{4},\ \frac{1}{2}n_1+\frac{1}{2}\\ \frac{1}{2}\lambda+n_1+\frac{1}{2}n+r+5/4\end{matrix};\ 1-\frac{1}{2p}\right\},$$

$$\mathbf{R}(\lambda+n+n_1+\tfrac{1}{2})>0,\ \mathbf{R}(p)>0 \text{ and } |\arg p|<\tfrac{1}{2}\pi.$$

Hence by Theorem II with $k = \frac{1}{2}n+l+\frac{1}{4}$ and $m = \pm\frac{1}{2}n$, we have

$$\int_0^{\infty} W_\lambda^2(p/x^2)\,e^{-x^2/4}\sum_{t=0}^{n_1}\frac{(-)^{n_1-t}n_1!}{t!\,[(n_1-t)!]^2}\,x^{n_1-t}D_{n_1-t}(x)dx$$

$$= \frac{2\sqrt{\pi}.(-)^l l!\,\Gamma(n+l+1)}{2^{3n_1/2+\frac{3}{2}+\lambda+\frac{1}{2}n}\,p^{\frac{1}{2}(\lambda+n_1)}}\sum_{r=0}^{\infty}\frac{(-)^r\Gamma(\lambda+2n+n_1+2r+\frac{1}{2})2^{-r}}{r!\,(l-r)!\,\Gamma(1+n+r)\Gamma(\frac{1}{2}\lambda+\frac{1}{2}n+r+n_1+5/4)}$$

$$\times\,{}_2F_1\left\{\begin{matrix}\frac{1}{2}n_1+\frac{1}{2}\lambda+\frac{1}{2}n+r+\frac{1}{4},\ \frac{1}{2}n_1+\frac{1}{2}\\ n_1+\frac{1}{2}\lambda+\frac{1}{2}n+r+5/4\end{matrix};\ 1-\frac{1}{2p}\right\},$$

$$\mathbf{R}(\lambda+n+n_1+\tfrac{1}{2})>0,\ \mathbf{R}(p)>0 \text{ and } |\arg p|<\tfrac{1}{2}\pi.$$

II

2.1. In this section I have established another theorem in a way reciprocal to Theorem II. This theorem is useful in evaluating integrals involving the new function $T(s)$ defined below.

2.2. Theorem III. If

$$\varphi(p) \doteqdot h(x)$$

and

$$p^{1-m_1}h(p) \overset{k}{\underset{m}{\doteqdot}} g(x)$$

then, povided the integral is convergent,

$$\varphi(p) = \int_0^{\infty} g(s)T(s)ds,$$

where

$$T(s) = \frac{p\Gamma(m_1+m+5/4)\Gamma(m_1-m+5/4)}{(2s)^{m_1+1}\Gamma(m_1-k+7/4)}\cdot{}_2F_1\left\{\begin{matrix}m_1+m+5/4,\ \ m_1-m+5/4\\ m_1-k+7/4\end{matrix};\ \frac{1}{2}-\frac{p}{2s}\right\}.$$

$\mathbf{R}(\mu\pm m+\frac{1}{4})>0$, $\mathbf{R}(m_1)>0$, $\mathbf{R}(m_1\pm m+5/4)>0$, $\mathbf{R}(p)>0$, $g(x) = O(x^{\mu})$ for small x and $g(x) = O(1)$ for large x.

2.3. *Proof of Theorem III.*

Here

$$\frac{1}{p}\varphi(p) = \int_0^{\infty} e^{-px}h(x)dx \tag{1}$$

and

$$x^{1-m_1}h(x) = x\int_0^\infty (2xs)^{-\frac{1}{4}}W_{k,m}(2xs)g(s)ds. \tag{2}$$

Substituting the value of $h(x)$ from (2) in (1), we have

$$\frac{1}{p}\varphi(p) = \int_0^\infty e^{-px}dx.x^{m_1}\int_0^\infty (2xs)^{-\frac{1}{4}}W_{k,m}(2xs)g(s)ds$$

$$= \int_0^\infty g(s)\left[\int_0^\infty x^{m_1}(2xs)^{-\frac{1}{4}}e^{-px}W_{k,m}(2xs)dx\right]ds. \tag{3}$$

Further (B., Integral D. 2), we get

$$\frac{1}{p}\varphi(p) = \frac{\Gamma(m_1+m+5/4)\Gamma(m_1-m+5/4)}{2^{m_1+1}\Gamma(m_1-k+7/4)}$$

$$\times\int_0^\infty s^{-m_1-1}g(s)\,{}_2F_1\left\{\begin{matrix}m_1+m+5/4, & m_1-m+5/4\\ & m_1-k+7/4\end{matrix};\ \tfrac{1}{2}-p/2s\right\}ds,$$

$\mathbf{R}(m_1\pm m+5/4)>0$ and $\mathbf{R}(p)>0$.

In order to justify the change in the order of integration in (3), suppose that

$$\theta(x) = g(x)\int_0^\infty x^{m_1}(2xs)^{-\frac{1}{4}}e^{-px}W_{k,m}(2xs)dx = g(x).T(x).$$

Now, I have shown below that

$$T(x) = O(x^{\pm m+\frac{1}{4}}) \text{ for small } x.$$

It is evident that $\theta(x)$ is uniformly convergent in $x \geqq 0$, if $\mathbf{R}(\mu\pm m+\frac{1}{4})>0$, where $g(x) = O(x^{\mu})$ for small x.

Again, consider the integral

$$I = \int_0^\infty g(s)\left[\int_A^\infty x^{m_1}(2xs)^{-\frac{1}{4}}e^{-px}W_{k,m}(2xs)dx\right]ds$$

where A is large.

Now, for large values of s, I does not exceed a constant multiple of

$$\int_0 |g(s)|\left[\int_A^\infty |(2xp)^{-\frac{1}{4}}x^{m_1}e^{-xp}W_{k,m}(2xs)|\,dx\right]ds$$

which tends to zero, if $\mathbf{R}(p)>0$, $\mathbf{R}(m_1)>0$ where $g(s)=O(1)$ for large s and $T(s) = O(s^{-m_1-1})$ for large s as shown below.

2.4. *Properties of $T(s)$.*

We have

$$T(s) = p\,\frac{\Gamma(m_1+m+5/4)\Gamma(m_1-m+5/4)}{2^{m_1+1}\Gamma(m_1-k+7/4)s^{m_1+1}}.{}_2F_1\left\{\begin{matrix}m_1+m+5/4, & m_1-m+5/4\\ & m_1-k+7/4\end{matrix};\ \tfrac{1}{2}-p/2s\right\}.$$

From the asymptotic expansion of ${}_2F_1$, we find that

$$\begin{aligned} T(s) &= O(s^{-m_1-1}) \text{ for large } s, \\ &= O(s^{\pm m+\frac{1}{2}}) \text{ for small } s. \end{aligned}$$

2.5. Corollary.

If in (2.2.1), we put $k = \frac{1}{4}$, $m = \pm\frac{1}{4}$, Whittaker Transform reduces to Laplace's transform and our theorem gives the result due to Shastri (1944).

We may further note that for $m_1 = 1$, we get the result due to Humbert (1937).

2.6. *Example.* Take

$$g(x) = x^{\alpha}e^{-x}$$

then, we* have

$$\frac{\Gamma(\alpha+m+5/4)\Gamma(\alpha-m+5/4)}{2(2p)^{\alpha}\Gamma(\alpha-k+7/4)} \cdot {}_2F_1\left\{\begin{matrix}\alpha+m+5/4,\ \alpha-m+5/4\\ \alpha-k+7/4\end{matrix};\ \tfrac{1}{2}-1/2p\right\} \overset{k}{\underset{m}{=}} x^{\alpha}e^{-x},$$

$$\mathbf{R}(\alpha\pm m+5/4)>0,\ |p|>1 \text{ and } \mathbf{R}(p)>0,$$

if $k = \frac{3}{4}$, $m = \pm\frac{1}{4}$, then

$$x^{1-m_1}h(x) = \frac{\Gamma(\alpha+3/2)x^{3/2}}{2^{-\frac{1}{2}}(1+x)^{\alpha+3/2}}$$

or

$$h(x) = \frac{\Gamma(\alpha+3/2)}{2^{-\frac{1}{2}}} \cdot \frac{x}{(1+x)^{\alpha+3/2}}, \text{ with } m_1 = \tfrac{1}{2}$$

$$\doteqdot \frac{\Gamma(\alpha+3/2)}{2^{-\frac{1}{2}}}\{1-(p-\alpha-\tfrac{1}{2})S(\alpha+3/2,\ p)\}\dagger$$

$$= \varphi(p).$$

Hence, from Theorem III, we get

$$\int_0^{\infty} s^{\alpha}e^{-s}T(s)ds = \frac{\Gamma(\alpha+3/2)}{2^{-\frac{1}{2}}}\{1-(p-\alpha-\tfrac{1}{2})S(\alpha+3/2,\ p)\},$$

$$\mathbf{R}(\alpha+3/2)>0,\ |p|>1 \text{ and } \mathbf{R}(p)>0.$$

$$T(s) = p\frac{\sqrt{(2s)}}{(p+s)^2}.$$

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* See B. result (1.2.3) with $n=\alpha$ and $q=1$.

† See H., p. 14.

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A STUDY OF THE GENERALISED LAPLACE INTEGRAL—III

By

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1.1. The purpose of this paper* is to obtain the Whittaker Transform of $h(x)$ when $x^{\lambda}h(x)$ is self-reciprocal in the Hankel transform of order ν. I have also considered the corresponding results in sine and cosine transforms. The results are pretty general and have been illustrated by examples. Some known results come out as particular cases.

1.2. Theorem IV. If

$$\varphi(p) \overset{k}{\underset{m}{=}} h(x)$$

and $x^{\lambda}h(x)$ is self-reciprocal in the Hankel transform of order ν, then, provided the integral is convergent

$$\varphi(p^{\mu}) = \int_0^{\infty} y^{\lambda+\frac{1}{4}} h(y) S_{\lambda}^{\mu}(y/p^{\mu})dy \tag{1}$$

where

$$S_{\lambda}^{\mu}(y/p^{\mu}) = \frac{p^{\mu(\lambda-\frac{1}{4})}}{2^{-\lambda+3/2}}\left(\frac{y}{4p^{\mu}}\right)^{\nu} \sum_{r=0}^{\infty} \frac{(-)^r\Gamma(\nu-\lambda+m+2r+7/4)}{r!\,\Gamma(\nu+r+1)}$$

$$\times\frac{\Gamma(\nu-\lambda-m+2r+7/4)}{\Gamma(\nu-\lambda-k+2r+9/4)}\left(\frac{1}{4}\cdot\frac{y}{p^{\mu}}\right)^{2r} {}_2F_1\left\{\begin{matrix}\nu-\lambda+m+2r+7/4, & \nu-\lambda-m+2r+7/4\\ & \nu-\lambda-k+2r+9/4\end{matrix}\,;\ \tfrac{1}{2}\right\},$$

$\mathbf{R}(\lambda+\mu_1+\nu+3/2)\geq 0$, $\mathbf{R}(\nu-\lambda\pm m+7/4)>0$, $\mathbf{R}(\mu_1\pm m+5/4)>0$, $h(x)=O(x^{\mu_1})$ for small x and $h(x)=O\{\exp(-x^{\lambda_1})\}$, $\mathbf{R}(\lambda_1)>0)$ for large x.

1.3. *Proof of the Theorem IV.* Since

$$\varphi(p) \overset{k}{\underset{m}{=}} h(x)$$

$$\varphi(p) = p\int_0^{\infty}(2xp)^{-\frac{1}{4}}W_{k,m}(2px)h(x)dx \tag{1}$$

and $x^{\lambda}h(x)$ is† R_{ν},

$$x^{\lambda}h(x) = \int_0^{\infty}(xy)^{\frac{1}{2}}J_{\nu}(xy)y^{\lambda}h(y)dy \tag{2}$$

J_{ν} denoting the Bessel function of order ν.

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* This is in continuation to my paper, Bose (1949), II.

† This notation is due to Hardy and Littlewood. As a particular case a function is said to be R_c or R_s, if it is self-reciprocal in the cosine or sine transform ($\nu = \pm\frac{1}{2}$) and finally that a function is skew R_{ν} if it is self-reciprocal to $-f(x)$.

If we use (2) in (1) after replacing p by p^{μ}, we get

$$\varphi(p^{\mu}) = p^{\mu}\int_0^{\infty}(2xp^{\mu})^{-\frac{1}{2}}W_{k,m}(2p^{\mu}x)dx\int_0^{\infty}x^{-\lambda}\sqrt{(xy)}\,.\,y^{\lambda}J_{\nu}(xy)h(y)dy$$

$$= \frac{p^{\mu\lambda-\mu\nu-\frac{1}{2}\mu}}{2^{2\nu-\lambda+3/2}\Gamma(\nu+1)}\int_0^{\infty}y^{\lambda+\nu+\frac{1}{2}}h(y)\left[\int_0^{\infty}x^{\nu-\lambda+\frac{1}{2}}W_{k,m}(x)\,.\,{}_0F_1\{1+\nu, -x^2y^2/(16p^{2\mu})\}dx\right]dy \quad (3)$$

on changing the order of integration.

Now, using the integral of Goldstein (1932), with $\alpha^2 = -\frac{1}{2}$, after changing the variable and constants of (3), we get

$$\varphi(p^{\mu}) = \frac{p^{\mu\lambda-\frac{1}{2}\mu}}{2^{-\lambda+3/2}}\int_0^{\infty}y^{\lambda+\frac{1}{2}}h(y)(y/4p^{\mu})^{\nu}\sum_{r=0}^{\infty}\frac{(-)^r\Gamma(\nu-\lambda+m+2r+7/4)\Gamma(\nu-\lambda-m+2r+7/4)}{r!\,\Gamma(\nu-\lambda-k+2r+9/4)\Gamma(\nu+r+1)}$$

$$\times(y/4p^{\mu})^{2r}\,.\,{}_2F_1\left\{\begin{matrix}\nu-\lambda+m+2r+7/4, & \nu-\lambda-m+2r+7/4\\ & \nu-\lambda-k+2r+9/4\end{matrix}\,;\ \tfrac{1}{2}\right\}dy$$

provided $\mathbf{R}(\nu-\lambda\pm m+7/4) > 0$.

Now, we have to establish the validity of changing the order of integration in (3).

Integral (1) is absolutely convergent, if $\mathbf{R}(\mu_1\pm m+5/4) > 0$, where $h(x) = O(x^{\mu_1})$ for small x. Also the integral (2) is absolutely convergent when $\mathbf{R}(\nu+\lambda+\mu_1+3/2) > 0$.

Since, (Buschnall, 1940 and 1941),

$${}_2F_1\left\{\begin{matrix}a+2r, & 2b+2r\\ & c+2r\end{matrix}\,;\ x\right\} < \frac{(c)_{2r}}{(b)_r(b)_r}(1-x)^{-2r}\,.\,{}_2F_1\left\{\begin{matrix}a, & 2b\\ & c\end{matrix}\,;\ x\right\},$$

where

$$(a)_r \equiv \Gamma(a+r)/\Gamma(a).$$

By applying the test given in Carslaw: Fourier series and integrals, we get that (1.2.1) is convergent keeping in view the results of the next article, under the additional condition

$$h(x) \sim \exp(-x^{\lambda_1}),\quad \mathbf{R}(\lambda_1) > 0.$$

Thus the change in the order of integration is justified under the conditions in the theorem.

1.4. *Properties of* $S_{\lambda}^{\mu}(x)$.

The behaviour of $S_{\lambda}^{\mu}(x)$ for small and large values of x can easily be investigated. We have, in fact

$$S_{\lambda}^{\mu}(x) = \frac{p^{\mu(\lambda-\frac{1}{2})}}{2^{-\lambda+3/2}}x^{\nu}\sum_{r=0}^{\infty}\frac{\Gamma(\nu-\lambda+m+2r+7/4)x^{2r}}{r!\,2^{-k-\nu+\lambda-2r-5/4}\Gamma(\nu+r+1)}$$

$$\times\frac{1}{\Gamma(-k+m+\frac{1}{2})}\int_0^1 t^{-k+m-\frac{1}{2}}(1-t)^{\nu-\lambda-m+2r+7/4}(1-\tfrac{1}{2}t)^{-(k-m+\frac{1}{2})}dt$$

by virtue of the relation (2)

$${}_1F_1[\alpha, \beta; \gamma; x] = (1-x)^{\gamma-\alpha-\beta}\,.\,{}_2F_1[\gamma-\alpha, \gamma-\beta; \gamma; x]$$

and (Baily, 1935)

$$F[a, b; c; z] = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt$$

which reduces to

$$S_\lambda^\mu(x) = K_1 x^\nu \int_0^1 t^{-k+m-\frac{1}{2}}(1-t)^{\nu-\lambda-m+7/4}(1-\tfrac{1}{2}t)^{-(k-m+\frac{1}{2})}$$

$$\times {}_2F_1\left\{\begin{matrix}\frac{1}{2}\nu-\frac{1}{2}\lambda+\frac{1}{2}m+7/8, \quad \frac{1}{2}\nu-\frac{1}{2}\lambda+\frac{1}{2}m+11/8\\ \nu+1\end{matrix}; 16x^2(1-t)^4\right\}dt$$

where K_1 is a constant independent of t.

It follows by the definition of $S_\lambda^\mu(x)$ that

$$S_\lambda^\mu(x) = O(x^\nu)$$

for small values of x, and from the asymptotic expansion of ${}_2F_1$, we have

$$S_\lambda^\mu(x) = O(x^{\lambda\pm m-7/4})$$

for large values of x.

1.5. Corollary.

When $k = \frac{1}{4}$, $m = \pm\frac{1}{4}$, Whittaker transform reduces to Laplace transform and the theorem reduces to:

If

$$\varphi(p) \doteqdot h(x)$$

and $x^\lambda h(x)$ is R_ν, then

$$\varphi(p^\mu) = \frac{p^{\mu(\lambda-\frac{1}{2})}\Gamma(\nu-\lambda+3/2)}{2^\nu\Gamma(\nu+1)}\int_0^\infty y^{\lambda+\frac{1}{2}}h(y)(y/p^\mu)^\nu . {}_2F_1\left\{\begin{matrix}\frac{1}{2}\nu-\frac{1}{2}\lambda+\frac{3}{4}, \ \frac{1}{2}\nu-\frac{1}{2}\lambda+5/4\\ \nu+1\end{matrix}; -y^2/p^{2\mu}\right\}dy$$

provided $\mathbf{R}(\nu-\lambda+3/2) > 0$, $\mathbf{R}(\nu+\lambda+\mu_1+3/2) > 0$ and $\mathbf{R}(\mu_1+1) > 0$.

1.6. *Particular cases.*

(*i*) If in (1.2.1), we put $\mu = 1$, $\lambda = 0$, we get the theorem of Tewari (1945).

(*ii*) If in (1.2.1), we put $\mu = \frac{1}{2}$, then

$$\varphi(\sqrt{p}) = \frac{1}{2(2p)^{-\lambda+\frac{1}{2}}}\int_0^\infty y^{\lambda+\frac{1}{2}}h(y)S_\lambda^{\frac{1}{2}}(y/\sqrt{p})dy$$

and in this, if we put $k = \frac{1}{4}$, $m = \pm\frac{1}{4}$, we get

$$\varphi(\sqrt{p}) = \frac{1}{(\sqrt{p})^{-\lambda+\frac{1}{2}}}\int_0^\infty y^{\lambda+\frac{1}{2}}h(y)(y/2\sqrt{p})^\nu\left[\sum_{r=0}^\infty \frac{(-)^r\Gamma(\nu-\lambda+2r+3/2)}{r!\ \Gamma(1+\nu+r)}(y/2\sqrt{p})^{2r}\right]dy.$$

On using

$$\frac{1}{p^n} \doteqdot \frac{x^n}{\Gamma(n+1)} \quad \text{and} \quad z^{m+\frac{1}{2}}e^{\frac{1}{2}z}.{}_1F_1\left[\begin{matrix}\frac{1}{2}+m+k\\ 1+2m\end{matrix}; -z\right] = M_{k,m}(z)$$

we get a result of Gupta (1945).

(*iii*) If in (1.2.1), we put $k = m+\frac{1}{2}$, we have $W_{k,m}(x) = e^{-x}x^{m+\frac{1}{2}}$. Hence (1.2.1) reduces to

$$p^{\mu}\int_{0}^{\infty}(2xp^{\mu})^{m+\frac{1}{4}}\exp(-p^{\mu}x).h(x)dx = \frac{\Gamma(\nu-\lambda+m+7/4)p^{\mu(\lambda-\nu-\frac{1}{2})}}{2^{\nu-m-\frac{1}{4}}}\int_{0}^{\infty}h(y)y^{\lambda+\nu+\frac{1}{2}}$$

$$\times {}_2F_1\left\{\begin{matrix}\frac{1}{2}\nu-\frac{1}{2}\lambda+\frac{1}{2}m+7/8, \quad \frac{1}{2}\nu-\frac{1}{2}\lambda+\frac{1}{2}m+11/8 \\ \nu+1\end{matrix}; -y^2/p^{2\mu}\right\}dy$$

provided $\mathbf{R}(\lambda+\mu_1+\nu+3/2) > 0$, $\mathbf{R}(\nu-\lambda\pm m+7/4) > 0$ and $\mathbf{R}(\mu_1\pm m+5/4) > 0$.

1.7. *Examples.*

(*i*) Take

$$x^{\lambda}h(x) = x^{\nu+\frac{1}{2}}e^{-\frac{1}{4}x^2}$$

we have, from Bose (1949) I*

$$\varphi(p^{\mu}) \equiv \frac{p^{\mu}\exp(\frac{1}{4}p^{2\mu})}{\Gamma(\frac{1}{2}-m-k)\,\Gamma(\frac{1}{2}+m-k)}\Big[\Gamma(-2m)\,\Gamma(2m+1)$$

$$\times\sum_{r=0}^{\infty}\frac{\Gamma(m+\frac{1}{2}-k+r)\Gamma(\nu-\lambda+m+r+7/4)(2p^{\mu})^{m+r+\frac{1}{4}}}{r!\,\Gamma(2m+1+r)}.D_{-(\nu-\lambda+m+r+7/4)}(p^{\mu})$$

$$+\Gamma(2m)\Gamma(-2m+1)\sum_{r=0}^{\infty}\frac{\Gamma(-m+\frac{1}{2}-k+r)\,\Gamma(\nu-\lambda-m+r+7/4)}{r!\,\Gamma(-2m+1+r)}$$

$$\times(2p^{\mu})^{-m+r+\frac{1}{4}}D_{-(\nu-\lambda-m+r+7/4)}(p^{\mu})\Big].$$

Hence from (1.2.1), we get

$$\int_{0}^{\infty}y^{\nu+1}e^{-\frac{1}{4}y^2}S^{\mu}_{\lambda}(y/p^{\mu})dy = \frac{p^{\mu}\exp(\frac{1}{4}p^{2\mu})}{\Gamma(\frac{1}{2}-m-k)\Gamma(\frac{1}{2}+m-k)}\Big[\Gamma(-2m)\Gamma(2m+1) \tag{1}$$

$$\times\sum_{r=0}^{\infty}\frac{\Gamma(m+\frac{1}{2}-k+r)\Gamma(\nu-\lambda+m+r+7/4)(2p^{\mu})^{m+r+\frac{1}{4}}}{r!\,\Gamma(r+2m+1)}\times D_{-(\nu-\lambda+m+r+7/4)}(p^{\mu})$$

$$+\Gamma(2m)\Gamma(-2m+1)\sum_{r=0}^{\infty}\frac{\Gamma(-m+\frac{1}{2}-k+r)\Gamma(\nu-\lambda-m+r+7/4)(2p^{\mu})^{-m+r+\frac{1}{4}}}{r!\,\Gamma(-2m+1+r)}$$

$$\times D_{-(\nu-\lambda-m+r+7/4)}(p^{\mu})\Big], \quad \mathbf{R}(\nu-\lambda\pm m+7/4) > 0 \text{ and } \mathbf{R}(\nu+1) > 0.$$

Also, with $x^{\lambda}h(x) = x^{\nu+\frac{1}{2}}e^{-\frac{1}{4}x^2}$ (1.2.1) with $k = 0$, may be written as,

$$p^{\mu}\int_{0}^{\infty}(2xp^{\mu})^{\frac{1}{2}}K_{m}(xp^{\mu})x^{\nu+\frac{1}{2}-\lambda}e^{-\frac{1}{4}x^2}dx = \frac{p^{\mu(\lambda-\nu-\frac{1}{2})}\,\Gamma(\frac{1}{2}\nu-\frac{1}{2}\lambda+\frac{1}{2}m+7/8)\Gamma(\frac{1}{2}\nu-\frac{1}{2}\lambda-\frac{1}{2}m+7/8)}{2^{\lambda}\Gamma(\nu+1)}$$

$$\times\int_{0}^{\infty}y^{2\nu+1}e^{-\frac{1}{4}y^2}.{}_2F_1\left\{\begin{matrix}\frac{1}{2}\nu-\frac{1}{2}\lambda+\frac{1}{2}m+7/8, \ \frac{1}{2}\nu-\frac{1}{2}\lambda-\frac{1}{2}m+7/8 \\ \nu+1\end{matrix}; -y^2/p^{2\mu}\right\}dy,$$

$$\mathbf{R}(\nu-\lambda\pm m+7/4) > 0 \text{ and } \mathbf{R}(\nu+1) > 0.$$

---

* See result (1.2.4).

This with the help of the integral of Meijer (1934)

$$W_{k,m}(\xi^2) = \frac{4\xi e^{-\frac{1}{2}\xi^2}}{\Gamma(\frac{1}{2}+m-k)\,\Gamma(\frac{1}{2}-m-k)}\int_0^\infty e^{-u^2}K_{2m}(2\xi u)u^{-2k}du,$$

$$\mathbf{R}(\tfrac{1}{2}\pm m-k)>0.$$

gives

$$2^{\frac{1}{2}\nu-\frac{1}{2}\lambda-5/8}(2p^\mu)^{\frac{1}{2}}\Gamma(\tfrac{1}{2}\nu-\tfrac{1}{2}\lambda+\tfrac{1}{2}m+7/8)\Gamma(\tfrac{1}{2}\nu-\tfrac{1}{2}\lambda-\tfrac{1}{2}m+7/8)\exp(\tfrac{1}{4}p^{2\mu}).W_{\frac{1}{2}\lambda-\frac{1}{2}\nu-3/8,\,\frac{1}{2}m}(\tfrac{1}{2}p^{2\mu})$$

$$= \frac{p^{\mu(\lambda-\nu-\frac{1}{2})}\Gamma(\frac{1}{2}\nu-\frac{1}{2}\lambda+\frac{1}{2}m+7/8)\Gamma(\frac{1}{2}\nu-\frac{1}{2}\lambda-\frac{1}{2}m+7/8)}{2^\lambda\Gamma(\nu+1)}\int_0^\infty y^{2\nu+1}e^{-\frac{1}{2}y^2}$$

$$\times\,{}_2F_1\left\{\begin{matrix}\frac{1}{2}\nu-\frac{1}{2}\lambda+\frac{1}{2}m+7/8,\ \frac{1}{2}\nu-\frac{1}{2}\lambda-\frac{1}{2}m+7/8\\ \nu+1\end{matrix};\ -y^2/p^{2\mu}\right\}dy, \qquad (2)$$

provided $\mathbf{R}(\lambda+\mu_1+\nu+3/2)>0$, and $\mathbf{R}(\nu-\lambda\pm m+7/4)>0$.

On putting $\mu=1$, $\lambda=0$, in the above result, we get the integral due to Meijer (1934)

$$W_{-\frac{1}{2}\nu-3/8,\,\frac{1}{2}m}(\tfrac{1}{2}p^2) = \frac{e^{-p^2/4}2^{-\frac{1}{2}\nu+3/8}}{p^{\nu+3/4}\Gamma(\nu+1)}\int_0^\infty y^{2\nu+1}e^{-\frac{1}{2}y^2}.{}_2F_1\left\{\begin{matrix}\frac{1}{2}\nu+\frac{1}{2}m+7/8,\ \frac{1}{2}\nu-\frac{1}{2}m+7/8\\ \nu+1\end{matrix};\ -\frac{y^2}{p^2}\right\}dy.$$

provided that $\mathbf{R}(\nu\pm m+7/8)>0$ and $\mathbf{R}(\nu)>-1$.

If we compare (1) for $k=0$ with (2), we incidentally get that

$$2^{\frac{1}{2}\nu-\frac{1}{2}\lambda-5/8}\Gamma(\tfrac{1}{2}m-\tfrac{1}{2}\lambda+\tfrac{1}{2}\nu+7/8)\Gamma(-\tfrac{1}{2}m-\tfrac{1}{2}\lambda+\tfrac{1}{2}\nu+7/8)W_{\left\{\begin{smallmatrix}\frac{1}{2}\lambda-\frac{1}{2}\nu-3/8\\ \frac{1}{2}m\end{smallmatrix}\right.}(p^{2\mu}/2)$$

$$= \sqrt{\pi}.p^\mu\Bigg[\frac{\Gamma(-2m)\Gamma(2m+1)}{\Gamma(\frac{1}{2}-m)\Gamma(\frac{1}{2}+m)}\sum_{r=0}^\infty\frac{\Gamma(r+m+\frac{1}{2})\Gamma(\nu-\lambda+m+r+7/4)}{r!\,\Gamma(r+2m+1)(2p^\mu)^{-m-r}}D_{-(\nu-\lambda+m+r+7/4)}(p^\mu)$$

$$+\frac{\Gamma(2m)\Gamma(-2m+1)}{\Gamma(\frac{1}{2}-m)\Gamma(\frac{1}{2}+m)}\sum_{r=0}^\infty\frac{\Gamma(r-m+\frac{1}{2})\,\Gamma(\nu-\lambda-m+r+7/4)}{r!\,\Gamma(r-2m+1)(2p^\mu)^{-r+m}}D_{-(\nu-\lambda-m+r+7/4)}(p^\mu)\Bigg].$$

1.8. An interesting particular case of Theorem IV is obtained on putting $\lambda=0$, $\nu=\frac{1}{2}$ and $\mu=1$: If

$$\varphi(p)\ \overset{k}{\underset{m}{=}}\ h(x),$$

then, provided the integral is convergent,

$$\varphi(p) = \frac{1}{4\sqrt{2}.p}\int_0^\infty yh(y)\sum_{r=0}^\infty\frac{\Gamma(m+2r+9/4)\,\Gamma(-m+2r+9/4)}{r!\,\Gamma(r+3/2)\Gamma(-k+2r+11/4)}\left(\frac{-y^2}{16p^2}\right)^r \qquad (1)$$

$$\times\,{}_2F_1\left\{\begin{matrix}m+2r+9/4,\quad -m+2r+9/4\\ -k+2r+11/4\end{matrix};\ \tfrac{1}{2}\right\}dy,\ \mathbf{R}(\mu_1+2)>0 \text{ and } \mathbf{R}(\pm m+9/4)>0$$

where $h(x)$ is* R_s

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* A function $h(x)$ which is R_s satisfies the integral equation

$$h(x) = \sqrt{(2/\pi)}\int_0^\infty h(y)\sin(xy)dy.$$

1.9. Corollary.

On putting $k = \frac{1}{4}$, $m = \pm \frac{1}{4}$, we get, provided the integral is convergent, the theorem of Mittal (1942).

1.10. *Examples.*

(*i*) The function $D_{2n+1}(x\sqrt{(2)}$ is $\pm R_s$ according as n is even or odd integer.

Using this function for n even, we have

$$\varphi(p) = \int_0^\infty (2xp)^{-\frac{1}{4}} W_{k,m}(2xp).D_{2n+1}(x\sqrt{2})dx$$

Now, by the help of [Bose (1949) I*] and the result due to Varma (1938-39)

$$D_{2n+1}(x) = \frac{\Gamma(-\frac{1}{2})2^n e^{-\frac{1}{4}x^2}}{\Gamma(-n-\frac{1}{2})}.{}_1F_1\{-n;\ 3/2;\ \tfrac{1}{2}x^2\}$$

and integrating by the integral of Bose (1949, I)† we get

$$\varphi(p) = \frac{e^{\frac{1}{4}r^2}p\Gamma(-\frac{1}{2})\Gamma(3/2)2^n}{\Gamma(-n)\Gamma(-n-\frac{1}{2})\Gamma(\frac{1}{2}-m-k)\,\Gamma(\frac{1}{2}+m-k)}\Bigg[\Gamma(-2m)\Gamma(2m+1)\sum_{r=0}^{\infty}\frac{\Gamma(r+m+\frac{1}{2}-k)}{r\,!\,\Gamma(r+2m+1)}$$

$$\times (2p)^{m+r+\frac{1}{4}}\sum_{r_1=0}^{n}\frac{\Gamma(r_1-n)\,\Gamma(m+r+2r_1+5/4)}{r_1\,!\,\Gamma(r_1+3/2)}D_{-(m+r+2r_1+5/4)}(p)+\Gamma(2m)\Gamma(-2m+1)$$

$$\times\sum_{r=0}^{\infty}\frac{\Gamma(r-m+\frac{1}{2}-k)}{r\,!\,\Gamma(r-2m+1)}(2p)^{-m+r+\frac{1}{4}}\sum_{r_1=0}^{n}\frac{\Gamma(r_1-n)\Gamma(-m+r+2r_1+5/4)}{r_1\,!\,\Gamma(r_1+3/2)}$$

$$\times D_{-(-m+r+2r_1+5/4)}(p)\Bigg],\quad \mathbf{R}(\pm m+5/4)>0.$$

Hence from (1.8.1), we have

$$\int_0^\infty y^{\frac{1}{4}}D_{2n+1}(y\sqrt{2})S_0^1(y/p)dy = \frac{e^{\frac{1}{4}r^2}p\Gamma(-\frac{1}{2})\Gamma(3/2)2^n}{\Gamma(-n)\Gamma(-n-\frac{1}{2})\Gamma(\frac{1}{2}-m-k)\,\Gamma(\frac{1}{2}+m-k)}\Bigg[\Gamma(-2m)\Gamma(2m+1)$$

$$\times\sum_{r=0}^{\infty}\frac{\Gamma(r+m+\frac{1}{2}-k)}{r\,!\,\Gamma(r+2m+1)}(2p)^{m+r+\frac{1}{4}}\sum_{r_1=0}^{n}\frac{\Gamma(r_1-n)\Gamma(m+r+2r_1+5/4)}{r\,!\,\Gamma(r_1+3/2)}D_{-(m+r+2r_1+5/4)}(p)$$

$$+\Gamma(2m)\Gamma(-2m+1)\sum_{r=0}^{\infty}\frac{\Gamma(r-m+\frac{1}{2}-k)}{r\,!\,\Gamma(r-2m+1)}(2p)^{-m+r+\frac{1}{4}}$$

$$\times\sum_{r_1=0}^{n}\frac{\Gamma(r_1-n)\,\Gamma(-m+r+2r_1+5/4)}{r_1\,!\,\Gamma(r_1+3/2)}D_{-(-m+r+2r_1+5/4)}(p)\Bigg]$$

provided $(2m)$ is not an integer and $\mathbf{R}(\pm m+5/4)>0$.

(*ii*) The function $xe^{-\frac{1}{4}x^2}$ is R_s.

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* See (D .1)

† See integral (1. 2, C),

Using this function, we obtain

$$\varphi(p) = \int_0^\infty y^{9/2} e^{-\frac14 y^2} S_0^1(y/p)\,dy.$$

On the other hand from Bose (1949, I)*, with $\lambda = 1$, we have

$$\varphi(p) = \frac{p e^{\frac14 p^2}}{\Gamma(\frac12 - m - k)\,\Gamma(\frac12 + m - k)} \Big[\Gamma(-2m)\Gamma(2m+1) \sum_{r=0}^{\infty} \frac{\Gamma(m + \frac12 - k + r)\Gamma(m + r + 9/4)}{r!\,\Gamma(2m+1+r)}$$

$$\times (2p)^{m+r+\frac14} D_{-(m+r+9/4)}(p) + \Gamma(2m)\Gamma(-2m+1) \sum_{r=0}^{\infty} \frac{\Gamma(-m+\frac12-k+r)\Gamma(-m+r+9/4)}{r!\,\Gamma(-2m+1+r)}$$

$$\times (2p)^{-m+r+\frac14} D_{-(-m+r+9/4)}(p) \Big]$$

provided $(2m)$ is not an integer and $\mathbf{R}(\pm m + 9/4) > 0$. Hence by (1. 8. 1), we get

$$\int_0^\infty y^{9/2} S_0^1(y/p) e^{-\frac14 y^2}\,dy = \frac{p\,e^{\frac14 p^2}}{\Gamma(\frac12 - m - k)\Gamma(\frac12 + m - k)} \Big[\Gamma(-2m)\Gamma(2m+1)$$

$$\times \sum_{r=0}^{\infty} \frac{\Gamma(r + m + \frac12 - k)\;\Gamma(m + r + 9/4)}{r!\;\Gamma(r + 2m + 1)} (2p)^{m+r+\frac14} D_{-(m+r+9/4)}(p) + \Gamma(2m)\Gamma(-2m+1)$$

$$\times \sum_{r=0}^{\infty} \frac{\Gamma(-m + \frac12 - k + r)\;\Gamma(-m + r + 9/4)}{r!\;\Gamma(-2m+1+r)} (2p)^{-m+r+\frac14} D_{-(-m+r+9/4)}(p) \Big]$$

provided $(2m)$ is not an integer and $\mathbf{R}(\pm m + 9/4) > 0$.

1.11. Another interesting particular case of Theorem IV is obtained on putting $\lambda = 0$, $\nu = -\frac12$ and $\mu = 1$. If

$$\varphi(p) \overset{k}{\underset{m}{=}} h(x),$$

then, provided the integral is convergent,

$$\varphi(p) = \frac{1}{\sqrt 2} \int_0^\infty h(y) \sum_{r=0}^{\infty} \frac{(-)^r \Gamma(m + 2r + 5/4)\,\Gamma(-m + 2r + 5/4)}{r!\;\Gamma(r + \frac12)\Gamma(-k + 2r + 7/4)} \left(\frac{y}{4p}\right)^{2r}$$

$$\times {}_2F_1\left\{ \begin{matrix} m + 2r + 5/4, \;\; -m + 2r + 5/4 \\ -k + 2r + 7/4 \end{matrix} ;\; \tfrac12 \right\} dy \quad (1)$$

where $h(x)$ is R_c.

1.12. *Examples*

(i) The function $e^{-\frac14 x^2}$ is R_c.

Using this function we have

$$\varphi(p) = \int_0^\infty e^{-\frac14 y^2} y^{\frac14} S_0^1(y/p)\,dy.$$

---

* See the result (1. 2 4).

On the other hand, from Bose (1949, I)*, with $\lambda = 0$, we have

$$\varphi(p) = \frac{p\, e^{\frac{1}{4}p^2}}{\Gamma(\frac{1}{2}-m-k)\Gamma(\frac{1}{2}+m-k)}\Big[\Gamma(-2m)\Gamma(2m+1)\sum_{r=0}^{\infty}\frac{\Gamma(m+\frac{1}{2}-k+r)\Gamma(m+r+5/4)}{r!\,\Gamma(2m+1+r)}$$

$$\times(2p)^{m+r+\frac{1}{4}}.D_{-(m+r+5/4)}(p)+\Gamma(2m)\Gamma(-2m+1)\sum_{r=0}^{\infty}\frac{\Gamma(-m+\frac{1}{2}-k+r)\Gamma(-m+r+5/4)}{r!\,\Gamma(-2m+1+r)}$$

$$\times(2p)^{-m+r+\frac{1}{4}}.D_{-(-m+r+5/4)}(p)\Big]$$

provided $(2m)$ is not an integer and $\mathbf{R}(\pm m+5/4)>0$, Hence by (1. 11. 1), we get

$$\int_0^{\infty} y^{\frac{1}{4}}e^{-\frac{1}{4}y^2}S_0^1(y/p)dy = \frac{p\, e^{\frac{1}{4}p^2}}{\Gamma(\frac{1}{2}-m-k)\Gamma(\frac{1}{2}+m-k)}\Big[\Gamma(-2m)\Gamma(2m+1)$$

$$\times\sum_{r=0}^{\infty}\frac{\Gamma(m+\frac{1}{2}-k+r)\Gamma(m+r+5/4)}{r!\,\Gamma(2m+1+r)}(2p)^{m+r+\frac{1}{4}}.D_{-(m+r+5/4)}(p)+\Gamma(2m)\Gamma(-2m+1)$$

$$\times\sum_{r=0}^{\infty}\frac{\Gamma(-m+\frac{1}{2}-k+r)\,\Gamma(-m+r+5/4)}{r!\,\Gamma(-2m+1+r)}(2p)^{-m+r+\frac{1}{4}}D_{-(-m+r+5/4)}(p)\Big]$$

provided $(2m)$ is not an integer and $\mathbf{R}(\pm m+5/4)>0$.

(*ii*) The function $D_{2n}(x\sqrt{2})$ is R_c for n an even integer.

Using this function, we have

$$\varphi(p) = p\int_0^{\infty}(2xp)^{-\frac{1}{4}}W_{k,m}(2px).D_{2n}(x\sqrt{2})dx$$

Now, by the help of Bose (1949, I)† and Varma (1938-39)

$$D_{2n}(x) = \frac{\Gamma(\frac{1}{2})2^n e^{-\frac{1}{4}x^2}}{\Gamma(-n+\frac{1}{2})}.{}_1F_1\{-n;\ \tfrac{1}{2};\ \tfrac{1}{2}x^2\}$$

and integrating by the help of Bose (1949, I)‡

$$\varphi(p) = \frac{2^n\pi p e^{\frac{1}{4}p^2}}{\Gamma(-n+\frac{1}{2})\Gamma(-n)\,\Gamma(\frac{1}{2}-m-k)\Gamma(\frac{1}{2}+m-k)}\Big[\Gamma(-2m)\Gamma(2m+1)$$

$$\times\sum_{r=0}^{\infty}\frac{\Gamma(r+m+\frac{1}{2}-k)\,(2p)^{m+r+\frac{1}{4}}}{r!\,\Gamma(r+2m+1)}\sum_{r_1=0}^{n}\frac{\Gamma(r_1-n)\Gamma(m+r+2r_1+5/4)}{r_1!\,\Gamma(r_1+\frac{1}{2})}$$

$$\times D_{-(m+r+2r_1+5/4)}(p)+\Gamma(2m)\Gamma(-2m+1)\sum_{r=0}^{\infty}\frac{\Gamma(r-m+\frac{1}{2}-k)(2p)^{-m+r+\frac{1}{4}}}{r!\,\Gamma(r-2m+1)}$$

$$\times\sum_{r_1=0}^{n}\frac{\Gamma(r_1-n)\Gamma(-m+r+2r_1+5/4)}{r_1!\,\Gamma(r_1+\frac{1}{2})}D_{-(-m+r+2r_1+5/4)}(p)\Big]$$

provided $(2m)$ is not an integer and $\mathbf{R}(\pm m+5/4)>0$.

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[*] See the result (1. 2. 4).

[†] See the result (D 1).

[‡] See the result (1. 2. C).

Hence, from (1. 11. 1), we have

$$\int_0^\infty y^{\frac{1}{4}} D_{2n}(y\sqrt{2}).S_0^1(y/p)\,dy = \frac{2^n \pi p e^{\frac{1}{4}p^2}}{\Gamma(-n+\frac{1}{2})\Gamma(-n)\Gamma(\frac{1}{2}-m-k)\Gamma(\frac{1}{2}+m-k)}\Bigg[\Gamma(-2m)\Gamma(2m+1)$$

$$\times \sum_{r=0}^{\infty} \frac{\Gamma(r+m+\frac{1}{2}-k)(2p)^{m+r+\frac{1}{4}}}{r!\,\Gamma(r+2m+1)} \sum_{r_1=0}^{n} \frac{\Gamma(r_1-n)\Gamma(m+r+2r_1+5/4)}{r_1!\,\Gamma(r_1+\frac{1}{2})} D_{-(m+r+2r_1+5/4)}(p)$$

$$+\Gamma(2m)\Gamma(-2m+1) \sum_{r=0}^{\infty} \frac{\Gamma(r-m+\frac{1}{2}-k)(2p)^{-m+r+\frac{1}{4}}}{r!\,\Gamma(r-2m+1)}$$

$$\times \sum_{r_1=0}^{n} \frac{\Gamma(r_1-n)\,\Gamma(-m+r+2r_1+5/4)}{r_1!\,\Gamma(r_1+\frac{1}{2})} D_{-(-m+r+2r_1+5/4)}(p)\Bigg]$$

provided $(2m)$ is not an integer and $\mathbf{R}(\pm m+5/4) > 0$.

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NOTE ON THE DEFORMATION PRODUCED BY SOME SYMMETRICAL DISTRIBUTION OF VARIABLE LOADS ON THE PLANE BOUNDARY OF A SEMI-INFINITE ELASTIC SOLID

BY

BIBHUTIBHUSAN SEN

(*Received May 16, 1949*)

INTRODUCTION

The problem of a load distributed *uniformly* over a circular area on the plane boundary of a semi-infinite elastic solid was solved by Boussinesq (1885), Lamb (1902), and Terazawa (1916a). A thorough discussion of this problem as well as that of uniformly distributed pressure over a rectangle on the plane boundary was given by Love (1929). Terazawa (1916b), and Love (1929) also considered the problem of a load varying as $(1-r^2/a^2)^{\frac{1}{2}}$ over a circular area of radius a on the plane surface of the solid, r being the distance from its centre. In the present note, an outline of a simple general method of solving problems of this type has been discussed at first. Then, the problem of the paraboloidal distribution, that is, the distribution of a load varying as $(1-r^2/a^2)$ over a circular area of radius a on the surface has been considered in detail. Finally, results for some distribution of loads varying exponentially on the plane boundary have been briefly stated.

1. METHOD OF SOLUTION

Suppose the plane boundary of an isotropic elastic solid is given by the plane $z = 0$, the axis of z being drawn into the solid. Then Beltrami's equtions give us

$$\nabla^2\widehat{xz}+\frac{1}{1+\sigma}\frac{\partial^2\odot}{\partial x\partial z}=0,\quad \nabla^2\widehat{yz}+\frac{1}{1+\sigma}\frac{\partial^2\odot}{\partial y\partial z}=0,$$

$$\nabla^2\widehat{zz}+\frac{1}{1+\sigma}\frac{\partial^2\odot}{\partial z^2}=0,\quad \nabla^2\odot=0. \tag{1.1}$$

where

$$\odot=\widehat{xx}+\widehat{yy}+\widehat{zz},\quad \nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2},$$

and σ is Poisson's ratio.

From the above equations, we have

$$\widehat{xz}=-\frac{1}{2(1+\sigma)}z\frac{\partial\odot}{\partial x}+\phi_1,$$

$$\widehat{yz}=-\frac{1}{2(1+\sigma)}z\frac{\partial\odot}{\partial y}+\phi_2,$$

$$\widehat{zz}=-\frac{1}{2(1+\sigma)}z\frac{\partial\odot}{\partial z}+\phi_3, \tag{1.2}$$

in which ϕ_1, ϕ_2, ϕ_3 are harmonic functions such that

$$[\phi_1]_{z=0} = [\widehat{xz}]_{z=0}, \quad [\phi_2]_{z=0} = [\widehat{yz}]_{z=0}, \quad [\phi_3]_{z=0} = [\widehat{zz}]_{z=0}. \tag{1.3}$$

In many particular cases it will be possible to determine these harmonic functions ϕ_1, ϕ_2 and ϕ_3 from the theory of potential. Substituting the values of $\widehat{xz}$, $\widehat{yz}$, $\widehat{zz}$ as obtained in (1.2) into the equation of equilibrium

$$\frac{\partial \widehat{xz}}{\partial x} + \frac{\partial \widehat{yz}}{\partial y} + \frac{\partial \widehat{zz}}{\partial z} = 0,$$

we get

$$\frac{\partial \odot}{\partial z} = 2(1+\sigma)\left[\frac{\partial \phi_1}{\partial x} + \frac{\partial \phi_2}{\partial y} + \frac{\partial \phi_3}{\partial z}\right], \tag{1.4}$$

whence $\odot$ is obtained by integration, the constant of integration being determined from the condition that $\odot$ tends to zero as z tends to infinity. Values of ϕ_1, ϕ_2, ϕ_3 and $\odot$ being known, the stress components $\widehat{xz}$, $\widehat{yz}$, $\widehat{zz}$ are completely determined. Then from the relations

$$\begin{aligned} \frac{\partial w}{\partial z} &= \frac{1}{E}\,[(1+\sigma)\widehat{zz} - \sigma\odot], \\ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} &= \frac{2(1+\sigma)}{E}\,\widehat{yz}, \\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} &= \frac{2(1+\sigma)}{E}\,\widehat{xz}, \end{aligned} \tag{1.5}$$

it will in general be possible to determine the components of displacement u, v w. if we impose the restriction that these components vanish when z becomes indefinitely large. The method enunciated above can be applied to the solution of problems of *symmetrical* as well as *non-symmetrical* distribution of loads on the plane boundary $z = 0$.

2. SOLUTION FOR SYMMETRICAL DISTRIBUTION OF LOADS ON THE SURFACE.

Suppose on the plane $z = 0$, $\widehat{xz} = \widehat{yz} = 0$, and

$$\widehat{zz} = -f(r) = -\int_0^\infty B(\alpha) J_0(\alpha r)\, d\alpha, \tag{2.1}$$

where $B(\alpha)$ is a function of α, and $J_0(\alpha r)$ is a Bessel's function of order zero, r being equal to $(x^2+y^2)^{\frac{1}{2}}$. Then in (1.2) we can put $\phi_1 = \phi_2 = 0$, and

$$\phi_3 = -\int_0^\infty B(\alpha) e^{-\alpha z} J_0(\alpha r)\, d\alpha. \tag{2.2}$$

The relation (1.4) is satisfied if we put

$$\odot = -2(1+\sigma)\int_0^\infty B(\alpha) e^{-\alpha z} J_0(\alpha r)\, d\alpha. \tag{2.3}$$

Substituting these results into (1.2), we get

$$\widehat{xz} = \int_0^\infty B(\alpha) z e^{-\alpha z} \frac{\partial}{\partial x}[J_0(\alpha r)]\,d\alpha,$$

$$\widehat{yz} = \int_0^\infty B(\alpha) z e^{-\alpha z} \frac{\partial}{\partial y}[J_0(\alpha r)]\,d\alpha,$$

$$\widehat{zz} = -\int_0^\infty B(\alpha)(\alpha z+1) e^{-\alpha z} J_0(\alpha r)\,d\alpha. \qquad (2.4)$$

Then using the equations (1.5), we get

$$u = \frac{1+\sigma}{E}\int_0^\infty \left\{\frac{B(\alpha)}{\alpha^2}[(1-2\sigma)-\alpha z]e^{-\alpha z}\frac{\partial}{\partial x}[J_0(\alpha r)]\right\}d\alpha,$$

$$v = \frac{1+\sigma}{E}\int_0^\infty \left\{\frac{B(\alpha)}{\alpha^2}[(1-2\sigma)-\alpha z]e^{-\alpha z}\frac{\partial}{\partial y}[J_0(\alpha r)]\right\}d\alpha,$$

$$w = \frac{1+\sigma}{E}\int_0^\infty \frac{B(\alpha)}{\alpha}[2(1-\sigma)+\alpha z]e^{-\alpha z}J_0(\alpha r)\,d\alpha. \qquad (2.5)$$

On the plane boundary $z = 0$,

$$w_0 = [w]_{z=0} = \frac{2(1-\sigma^2)}{E}\int_0^\infty \frac{B(\alpha)}{\alpha}J_0(\alpha r)\,d\alpha. \qquad (2.6)$$

$$\odot_0 = [\odot]_{z=0} = -2(1+\sigma)\int_0^\infty B(\alpha)J_0(\alpha r)\,d\alpha.$$

Denoting by $\widehat{rr}_0$ and $\widehat{\theta\theta}_0$ the surface values of $\widehat{rr}$ and $\widehat{\theta\theta}$, we get

$$\begin{aligned}\widehat{rr}_0+\widehat{\theta\theta}_0 &= \odot_0-\widehat{zz}_0\\ &= -(1+2\sigma)\int_0^\infty B(\alpha)J_0(\alpha r)\,d\alpha \qquad (2.7))\\ &= (1+2\sigma)\widehat{zz}_0\end{aligned}$$

Again since

$$u_r = \frac{x}{r}u+\frac{y}{r}v,$$

we have from (2.5)

$$u_r = \frac{1+\sigma}{E}\int_0^\infty \frac{B(\alpha)}{\alpha^2}[(1-2\sigma)-\alpha z]e^{-\alpha z}\frac{\partial}{\partial r}[J_0(\alpha r)]\,d\alpha,$$

so that on the surface $z = 0$,

$$\widehat{rr}_0 - \widehat{\theta\theta}_0 = 2\mu\left[\frac{\partial u_r}{\partial r} - \frac{u_r}{r}\right]_{z=0}$$

$$= (1-2\sigma)\int_0^\infty \frac{B(\alpha)}{\alpha^2}\left[\frac{\partial^2}{\partial r^2}J_0(\alpha r) - \frac{1}{r}\frac{\partial}{\partial r}J_0(\alpha r)\right]d\alpha$$

$$= (1-2\sigma)\int_0^\infty B(\alpha)J_2(\alpha r)d\alpha. \tag{2.8}$$

Also

$$[\widehat{zz}]_{r\to 0} = -\int_0^\infty B(\alpha)(\alpha z+1)e^{-\alpha z}d\alpha \tag{2.9}$$

3. NORMAL LOAD ON THE PLANE SURFACE VARYING AS $(1-r^2/a^2)$ OVER A CIRCULAR AREA

On the surface $z = 0$, let $\widehat{xz} = \widehat{yz} = 0$, and

$$\begin{aligned}\widehat{zz} &= -f(r) = -p(1-r^2/a^2), \quad r < a. \\ &= 0 \qquad\qquad\qquad\qquad\quad , \quad r \geq a,\end{aligned} \tag{3.1}$$

where p is a constant.

Since we can put

$$f(r) = \int_0^\infty J_0(\alpha r)\alpha d\alpha \int_0^\infty f(\lambda)J_0(\alpha\lambda)\lambda d\lambda, \tag{3.2}$$

we get in this case

$$f(r) = p\int_0^\infty J_0(\alpha r)\alpha d\alpha \int_0^a (1-\lambda^2/a^2)J_0(\alpha\lambda)\lambda d\lambda. \tag{3.3}$$

Putting $\alpha\lambda = z_1$ and $\alpha a = a_1$, we have the integral

$$\int_0^a \left(1-\frac{\lambda^2}{a^2}\right)J_0(\alpha\lambda)d\lambda = \frac{1}{\alpha^2}\int_0^{a_1}\left(1-\frac{z_1^2}{a_1^2}\right)J_0(z_1)z_1 dz_1$$

$$= \frac{1}{\alpha^2}\int_0^{a_1}\left(1-\frac{z_1^2}{a_1^2}\right)\frac{d}{dz_1}[z_1J_1(z_1)]dz_1.$$

On integration by parts, the above integral reduces to

$$\frac{2}{\alpha^2 a_1^2}\int_0^{a_1} z_1^2 J_1(z_1)dz_1 = \frac{2}{\alpha^2 a_1^2}\int_0^{a_1}\frac{d}{dz}[z_1^2 J_1(z_1)]dz_1 = \frac{2}{\alpha^3}J_2(a\alpha).$$

Hence

$$f(r) = 2p\int_0^\infty J_0(\alpha r)J_2(\alpha a)\frac{d\alpha}{\alpha},$$

so that from (2.1), we get

$$B(\alpha) = \frac{2pJ_2(\alpha a)}{\alpha}. \tag{3.4}$$

Then using the result given in (2.6), we obtain :

$$w_0 = \frac{4(1-\sigma^2)p}{E}\int_0^\infty \frac{J_2(\alpha a)J_0(\alpha r)}{\alpha^2}\,d\alpha$$

$$= \frac{4(1-\sigma^2)pa}{3E}\,{}_2F_1\left[\frac{1}{2},\ -\frac{3}{2}\ ;\ 1\ ;\ \frac{r^2}{a^2}\right], \tag{3.5}$$

when $0 < r < a$ (Watson, 1944, p. 401).

Within the circular area w_0 has a maximum value when $r = 0$. If we denote this value by w_m, we get

$$w_m = \frac{4(1-\sigma^2)}{3E}\,pa.$$

P, the magnitude of the total pressure is given by

$$P = \int_0^a p\left(1-\frac{r^2}{a^2}\right)2\pi r dr = \frac{\pi p a^2}{2}.$$

Hence

$$w_m = \frac{8(1-\sigma^2)P}{3\pi E a}. \tag{3.6}$$

From (2.7) and (3.1) we get

$$\begin{aligned} \widehat{rr}_0 + \widehat{\theta\theta}_0 &= -(1+2\sigma)p(1-r^2/a^2), & 0 < r < a \\ &= 0 & , \quad r \geq a. \end{aligned} \tag{3.7}$$

Substituting the value of $B(\alpha)$ obtained in (3.4) into (2.8), we have

$$\widehat{rr}_0 - \widehat{\theta\theta}_0 = 2(1-2\sigma)p\int_0^\infty \frac{J_2(\alpha r)J_2(\alpha a)}{\alpha}\,d\alpha$$

$$= \frac{(1-2\sigma)}{2}\,p\,\frac{r^2}{a^2}, \qquad 0 < r < a$$

$$= \frac{(1-2\sigma)}{2}\,p\,\frac{a^2}{r^2}, \qquad a \leq r \tag{3.8}$$

(Watson, 1944, p 405).

Hence

$$\widehat{rr}_0 = -\frac{p}{4}\left[2(1+2\sigma)-(3+2\sigma)\frac{r^2}{a^2}\right], \quad 0 < r < a,$$

$$= \frac{p}{4}\,(1-2\sigma)\frac{a^2}{r^2}, \qquad a \leq r. \tag{3.9}$$

$$\widehat{\theta\theta}_0 = -\frac{p}{4}\left[2(1+2\sigma)-(1+6\sigma)\frac{r^2}{a^2}\right], \quad 0 < r < a,$$

$$= -\frac{p}{4}\,(1-2\sigma)\frac{a^2}{r^2}, \qquad a \leq r. \tag{3.10}$$

4. Additional results

I. Let the normal load on the boundary $z = 0$ be given by

$$\widehat{zz}_0 = -p_0 e^{-h^2r^2}, \tag{4.1}$$

where p_0 and h are constants. Then we shall have

$$w_0 = \frac{(1-\sigma^2)\sqrt{\pi}.p_0}{Eh} e^{-\frac{1}{2}h^2r^2} I_0\left(\frac{h^2r^2}{2}\right), \tag{4.2}$$

and

$$w_m = \frac{Ph(1-\sigma^2)}{E.\sqrt{\pi}}, \tag{4.3}$$

P being the total pressure on the surface.

II. If the normal load on the boundary $z = 0$ be given as

$$\widehat{zz}_0 = -p_0 e^{-kr}, \tag{4.4}$$

where p_0 and k are constants, we have

$$w_0 = \frac{2(1-\sigma^2)p_0 k}{E} \int_0^\infty \frac{J_0(\alpha r)}{(\alpha^2+k^2)^{3/2}}\, d\alpha \tag{4.5}$$

and

$$w_m = \frac{P(1-\sigma^2)k}{E\pi}. \tag{4.6}$$

P denoting as before, the total pressure on the surface.

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A PROPERTY OF CLOSED REGIONS

By

D. C. Misra*

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Theorem. *Of all closed and bounded regions, each of which satisfies the property that the distance between any two points inside it is less than or equal to $2r$, the circular region of radius r has got the maximum area.*

Definition. A convex region R is defined to be a bounded and closed region with the property that if A and B are any two points belonging to R, then all points of the segment AB belong to R.

If R' is an arbitrary closed and bounded set, the curve C being the boundary, there is one "least convex envelope R of R' which is itself convex, which contains all points of R' and which consists of convex arcs of C and rectilinear portions of tangents to C which touch C at two points and bridge over the concave parts of C by straight lines. (Courant, 1936, pp. 100,494).

Lemma 1. *If R' is an arbitrary closed and bounded region with the property that the distance between any two points inside R' is less than or equal to $2r$, then the region R bounded by the "least convex envelope" of R' has the same property.*

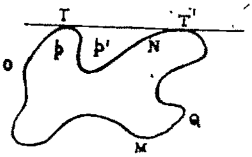

Fig. 1

Proof. Let $QMOTNT'$ be the arbitrary closed and bounded region R'. Let TT' be a tangent that bridges over the concave portion TNT' of the boundary C' of R' in the sense of definition (2). Call the region $QMOTT'$, R_1.

Now if we take any two points p, p' inside R_1, then following are the possiblities of their locations.

(*i*) Both p, p' lie in R'.

(*ii*) One of them lies in R', the other in R_1 but not in R'.

(*iii*) Both lie in R_1 but not in R'.

Case (*i*) Here the distance pp' is less than or equal to $2r$ by hypothesis.

Case (*ii*) Here either (a) $pp' \| TT'$ in which case pp' cuts C', so that $pp' \leq 2r$, or (b) pp' intersects TT' in which case $pp' \leq$ either pT or pT' or both, so that $pp' \leq 2r$; or cuts TT' extended in which case obviously $pp' \leq 2r$.

Case (*iii*) Here also either $pp' \| TT'$ or pp' intersects TT'; and the argument is similar as in case (*ii*); only here we replace p by a point on pp' inside R'.

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* Editorial board regrets to report that after the submission of this paper the author was involved in a boat accident under tragic circumstances and died.

Hence R_1 is a region which satisfies the property that the distance between any two points within it is less than or equal to $2r$. Then bridge over a concavity of R_1 and call the newly obtained region R_2. A similar argument holds in case of R_2 also. Thus proceeding we get at the least convex envelope, K of R' which satisfies the required property.

Again the area of the "least convex region" R bounded by K is always greater than or equal to that of R'. Hence our theorem is true for all regions, if it is true for convex regions.

Lemma 2. *If an arbitrary closed and bounded region satisfies the property that the distance between any two points inside it is less than or equal to $2r$ and if we take the origin O inside the region then the length of any straight line pp' passing through the origin and intercepted between two tangents, each to the boundary of R, and each orthogonal to pp', is less than or equal to $2r$.*

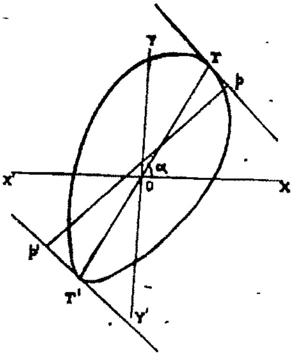

FIG. 2

Proof. The tangents $T'p'$ and Tp are parallel. Therefore $pp' \leq TT'$, *i.e.*, $\leq 2r$.

Proof of the theorem. Following the lines of proof of the isoperimetric problem (for convex curves) as indicated by Courant (1936, p. 213) let the convex region R be defined by the envelope of the family of straight lines (taking origin inside the region)

$$x \cos \alpha + y \sin \alpha - p(\alpha) = 0 \tag{A}$$

where $p(\alpha)$ is a periodic function of period 2π which is twice continuously differentiable. So the region R is given by (A) and (B) where (B) is

$$-x \sin \alpha + y \cos \alpha - p'(\alpha) = 0,$$

i.e., the envelope is

$$\left. \begin{aligned} x &= p \cos \alpha - p' \sin \alpha \\ y &= p \sin \alpha + p' \cos \alpha \end{aligned} \right\} \tag{C}$$

α being the parameter.

So

$$\begin{aligned} x' &= -(p+p'') \sin \alpha \\ y' &= (p+p'') \cos \alpha. \end{aligned}$$

Thus the area A of R is

$$A = \frac{1}{2}\int_0^{2\pi}(xy'-x'y)d\alpha = \frac{1}{2}\int_0^{2\pi}(p+p'')p\,d\alpha = \frac{1}{2}\int_0^{2\pi}(p^2-p'^2)d\alpha.$$

Now let

$$p(\alpha) = \tfrac{1}{2}a_0 + \sum_{\nu=1}^{\infty}(a_\nu \cos\nu\alpha + b_\nu \sin\nu\alpha),$$

so that

$$p'(\alpha) = \sum_{\nu=1}^{\infty}\nu(b_\nu \cos\nu\alpha - a_\nu \sin\nu\alpha),$$

so that

$$A = \tfrac{1}{2}\pi[\tfrac{1}{2}a_0^2 - \sum_{\nu=2}^{\infty}(\nu^2-1)(a_\nu^2+b_\nu^2)] \leq \tfrac{1}{4}\pi a_0^2.$$

But

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{2\pi} p(\alpha)d\alpha \\ &= \frac{1}{\pi}\left[\int_0^{\pi} p(\alpha)d\alpha + \int_\pi^{2\pi} p(\alpha)d\alpha\right] \\ &= \frac{1}{\pi}\left[\int_0^{\pi}\{p(\alpha)d\alpha + p(\alpha+\pi)d\alpha\}\right] \\ &= \frac{1}{\pi}\int_0^{\pi}[p(\alpha)+p(\alpha+\pi)]d\alpha. \end{aligned}$$

But

$$p(\alpha)+p(\alpha+\pi) \leq 2r \qquad \text{by lemma 2.}$$

Therefore

$$a_0 \leq 2r.$$

Thus

$$A \leq \frac{4r^2\pi}{4} \quad i.e., \quad \leq \pi r^2.$$

But when R is a circle of radius r, $A = \pi r^2$.

Hence the theorem is proved.

This is, in fact, a problem that arose in connection with our work in Statistics. Thanks are due to Dr. Rao for his framing the problem and to Messrs. D. B. Lahiri and M. N. Ghose for their valuable criticisms.

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ON THE CONVERGENCE OF CERTAIN LACUNARY TRIGONOMETRIC SERIES

By

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(Communicated by the Secretary—Received April 22, 1949)

The object of this note is to verify a statement made by A. Zygmund (1930) in his paper "On the convergence of lacunary trigonometric series." His statement is, it is impossible to find everywhere divergent series of the form

$$\sum_{k=1}^{\infty}(a_k \cos n_k\theta + b_k \sin n_k\theta)$$

where $n_1 < n_2 < \ldots$ are integers, $n_{k+1}/n_k > q > 1$; and $a_k, b_k \to 0$.

He considered the following numerical example

$$\sum_{k=1}^{\infty}\frac{\sin 10^k\theta}{\sqrt{k}}.$$

Let us consider the following series

$$\sum_{k=1}^{\infty}\left[\cos\frac{2}{3}\pi.\frac{\sin 10^k\pi x}{\sqrt{k}} - \sin\frac{2}{3}\pi.\frac{\cos 10^k\pi x}{\sqrt{k}}\right]$$

It is evident that in the interval (0, 1) all the irrational numbers are points of divergence.

At $x = \frac{1}{3}$ the series becomes

$$\sin\frac{\pi}{3}\left(1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots\right) = +\infty.$$

Now the series is equal to

$$\sum_{k=1}^{\infty}\frac{\sin(10^k\pi x - \frac{2}{3}\pi)}{\sqrt{k}}$$

Hence the series converges at $x = \frac{2}{3} + \cdot 33$, $\frac{2}{3} + \cdot 333$, $\frac{2}{3} + \cdot 33329$ etc.

It is clear that at each of the points considered above the series converges to different values.

Because the value of the series lies between 0 and ∞ we can see that the series takes up any and every value preassigned.

We can extend the result to any interval.

Hence the series

(1) diverges almost everywhere,
(2) converges almost everywhere,
(3) can take any preassigned value.

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ON A GENERALISATION OF WEBER'S PARABOLIC-CYLINDER FUNCTIONS

By

S. C. Mitra and A. Sharma

(*Received May 28, 1949*)

1. Let $D^{(k)}_{(k)}$ stand for the operator $\dfrac{d}{dy}\dfrac{1}{y^{k-2}}\dfrac{d}{dy}$ and $D^{(kn)}_{(k)}$ denote the operator $D^{(k)}_{(k)}$ operating on a function n times successively. Further let

$$\bar{D}_{km}(y) = e^{y^k/2k} D^{(km)}_{(k)}(e^{-y^k/k}) \tag{1}$$

and

$$\bar{D}_{km+1}(y) = -e^{y^k/2k} D^{(km+k-1)}_{(k)}(e^{-y^k/k}) \tag{2}$$

where $D^{(km+1)}_{(k)}$ stands for the operator $\dfrac{d}{dy} D^{(km)}_{(k)}$.

In an earlier paper (Mitra and Sharma, 1949), we have proved that $\bar{D}_{km}(y)$ and $\bar{D}_{km+1}(y)$ satisfy the orthogonal properties

$$\begin{aligned}\int_{-\infty}^{\infty} \bar{D}_{km}(y)\bar{D}_{kr}(y)dy &= 0, \quad (r \neq m)\\ &= k^{2m-1+1/k}\Gamma(m+1)\Gamma(m+1/k), \quad (r = m)\end{aligned} \tag{3}$$

and

$$\begin{aligned}\int_{-\infty}^{\infty} y^{k-2}\bar{D}_{km+1}(y)\bar{D}_{kr+1}(y) &= 0, \quad r \neq m\\ &= k^{2m+1/k}\Gamma(m+1)\Gamma(m+1+1/k), \quad (r = m).\end{aligned} \tag{4}$$

Further we obtained the recurrence-relations:

$$\bar{D}'_{km}(y) + \tfrac{1}{2}y^{k-1}\bar{D}_{km}(y) - km\, y^{k-2}\bar{D}_{km-k+1}(y) = 0 \tag{5}$$

$$\bar{D}_{km}(y) - y^{k-1}\bar{D}_{km-k+1}(y) + (km-k+1)\bar{D}_{km-k}(y) = 0 \tag{6}$$

$$\bar{D}_{km+1}(y) - y\bar{D}_{km}(y) + km\bar{D}_{km-k+1}(y) = 0 \tag{7}$$

$$\bar{D}'_{km+1}(y) + \tfrac{1}{2}y^{k-1}\bar{D}_{km+1}(y) - (km+1)\bar{D}_{km}(y) = 0 \tag{8}$$

and the formulae

$$\begin{aligned}\bar{D}_{km}(y) = e^{-y^k/2k}\Big[y^{km} - \frac{km(km-k+1)}{1!\,k}\,y^{km-k}&\\ + \frac{km(km-k)(km-k+1)(km-2k+1)}{2!\,k^2}\,y^{km-2k} - \ldots\Big]&\end{aligned} \tag{9}$$

and

$$\bar{D}_{km+1}(y) = e^{-y^k/2k}\left[y^{km+1} - \frac{(km+1)km}{1!\,k}\,y^{km-k+1} + \ldots\right] \tag{10}$$

Let us denote the polynomial parts by $H_{km}(y)$ and $H_{km+1}(y)$. We shall in the present note first obtain the generating functions of $H_{km}(y)$ and $H_{km+1}(y)$ and then the integral representation and a self-reciprocal property of $\bar{D}_{km}(y)$. Needless to say that all the properties of Weber's parabolic cylinder functions follow if we put $k = 2$.

2. GENERATING FUNCTIONS OF THE POLYNOMIALS $H_{km}(y)$ AND $H_{km+1}(y)$

The function

$$\bar{D}_{km}(y) = (-1)^m \frac{k^m \Gamma(m+1/k) e^{-y^k/2k}}{\Gamma(1/k)} \times {}_1F_1(-m;\, 1/k;\, y^k/k)$$

Let

$$h(w) = H_0(x) + \frac{H_1(x)w}{1} + \frac{H_k(x)w^k}{1.k} + \frac{H_{k+1}(x)w^{k+1}}{1.k(k+1)} + \cdots$$

$$+ \frac{H_{km}(x).w^{km}}{1.k(k+1)2k(2k+1)\cdots(km-k+1)km} + \cdots$$

$$= h_1(w) + h_2(w),$$

where

$$h_1(w) = H_0(x) + \frac{H_k(x)w^k}{1.k} + \frac{H_{2k}(x)w^{2k}}{1.k(k+1)2k} + \cdots$$

$$h_2(w) = \frac{H_1(x)w}{1} + \frac{H_{k+1}(x)w^{k+1}}{1.k(k+1)} + \cdots$$

Differentiating $h_2(w)$ with respect to w and making use of the recurrence-relation

$$H_{km+1}(x) = xH_{km}(x) - kmH_{km-k+1}(x),$$

we get

$$h_2'(w) = xH_0(x) + \left\{\frac{xH_k(x)w^k}{1.k} - \frac{H_1(x)w^k}{1}\right\} + \left\{\frac{xH_{2k}(x)w^{2k}}{1.k(k+1)2k} - \frac{H_{k+1}(x)w^{2k}}{1.k(k+1)}\right\} + \cdots$$

$$= x\left\{H_0(x) + \frac{H_k(x)w^k}{1.k} + \frac{H_{2k}(x)w^{2k}}{1.k(k+1)2k} + \cdots\right\}$$

$$- w^{k-1}\left\{H_1(x)w + \frac{H_{k+1}(x)w^{k+1}}{1.k(k+1)} + \cdots\right\}$$

$$= xh_1(w) - w^{k-1}h_2(w). \tag{11}$$

In a similar manner we can prove that

$$h_1'(w) = x^{k-1}w^{k-2}h_2(w) - w^{k-1}h_1(w). \tag{12}$$

Differentiating (11) once we get

$$h_2''(w) + 2w^{k-1}h_3(w) + w^{k-2}(w^k + k - 1 - x^k)h_2(w) = 0 \tag{13}$$

Let us write

$$h_3(w) = e^{-w^k/k}y,$$

then

$$h_2'(w) = e^{-w^k/k}\left(\frac{dy}{dw} - w^{k-1}y\right).$$

Substituting in the above equation, we get on simplification

$$\frac{d^2y}{dw^2} - x^k w^{k-2} y = 0,$$

the solutions of which are

$$ {}_0F_1\left(1 - \frac{1}{k}\ ;\ \frac{x^k w^k}{k^2}\right) \quad \text{and} \quad w \,.\, {}_0F_1\left(1 + \frac{1}{k}\ ;\ \frac{x^k w^k}{k^2}\right).$$

On comparing with the expression that we have assumed for $h_1(w)$ and $h_2(w)$, we find that

$$h_1(w) = \frac{d}{dw}\left[w \,.\, {}_0F_1\left(1 + \frac{1}{k}\ ;\ \frac{x^k w^k}{k^2}\right)\right] e^{-w^2/k}$$

and

$$h_2(w) = xw \,.\, {}_0F_1\left(1 + \frac{1}{k}\ ;\ \frac{x^k w^k}{k^2}\right) e^{-w^2/k}.$$

We therefore have

$$h_1(w) = {}_0F_1\left(\frac{1}{k}\ ;\ \frac{x^k w^k}{k^2}\right) e^{-w^2/k} = \sum_{m=0}^{\infty} \frac{\Gamma(1/k+1) H_{km}(x) w^{km}}{k^{2m-1}\Gamma(m+1)\Gamma(m+1/k)}, \tag{15}$$

and

$$h_2(w) = xw \,.\, {}_0F_1\left(1 + \frac{1}{k}\ ;\ \frac{x^k w^k}{k^2}\right) e^{-w^2/k} = \sum_{m=0}^{\infty} \frac{\Gamma(1/k+1) H_{km+1}(x) w^{km+1}}{k^{2m}\Gamma(m+1)\Gamma(m+1+1/k)}. \tag{16}$$

Putting $k = 2$ and $w = \pm\sqrt{2}$ in succession, we fall back upon the following results of Goldstein (1932), *viz.*,

$$\sum_{m=0}^{\infty} \frac{D_{2m}(x) 2^m}{(2m)\,!} = e^{-\frac{1}{4}x^2-1} \cosh (2^{\frac{1}{2}}x), \tag{17}$$

$$\sum_{m=0}^{\infty} \frac{D_{2m}(x) (-2)^m}{(2m)\,!} = e^{-\frac{1}{4}x^2+1} \cos (2^{\frac{1}{2}}x), \tag{17'}$$

$$\sum_{m=0}^{\infty} \frac{D_{2m+1}(x) 2^m}{(2m+1)\,!} = 2^{-\frac{1}{2}} e^{-\frac{1}{4}x^2-1} \sinh (2^{\frac{1}{2}}x), \tag{18}$$

$$\sum_{m=0}^{\infty} \frac{D_{2m+1}(x) (-2)^m}{(2m+1)\,!} = 2^{-\frac{1}{2}} e^{-\frac{1}{4}x^2+1} \sin (2^{\frac{1}{2}}x), \tag{18'}$$

after multiplying both sides by the factor $e^{-\frac{1}{4}x^2}$.

3. We can prove by direct term-by-term integration that

$$e^{-x^2/k} = \frac{k}{\Gamma(1/k)} \int_0^{\infty} e^{-u^k} {}_0F_1\left(\frac{1}{k}\ ;\ -\frac{x^k u^k}{k}\right) du.$$

Operating on both sides by the operator $D_{(k)}^{(km)}$, we find that

$$D_{(k)}^{(km)}(e^{-x^2/k}) = \frac{(-1)^m k^{m+1}}{\Gamma(1/k)} \int_0^{\infty} e^{-u^k} u^{mk} {}_0F_1\left(\frac{1}{k}\ ;\ -\frac{u^k x^k}{k}\right) du. \tag{19}$$

Putting $k = 2$ and using (1), we find that

$$e^{-x^k/2k}\bar{D}_{km}(x) = \frac{(-1)^m k^{m+1}}{\Gamma(1/k)}\int_0^\infty e^{-u^k} u^{mk}\,{}_0F_1\left(\frac{1}{k}\;;\;-\frac{u^k x^k}{k}\right)du. \tag{20}$$

Let us write $u^k = kv^k$. We then have

$$e^{-x^k/2k}\bar{D}_{km}(x) = \frac{(-1)^m k^{2m+1+1/k}}{\Gamma(1/k)}\int_0^\infty e^{-kv^k} v^{mk}\,{}_0F_1\left(\frac{1}{k}\;;\;-v^k x^k\right)dv$$

which reduces to Adamoff's (1906) result, on putting $k = 2$.

4. Let us now write $u^k/2k$ for u^k in (20). We find that

$$e^{-x^k/2k}\bar{D}_{km}(x) = \frac{(-1)^m k^{1-1/k}}{2^{m+1/k}\Gamma(1/k)}\int_0^\infty e^{-u^k/2k} u^{mk}\,{}_0F_1\left(\frac{1}{k}\;;\;-\frac{u^k x^k}{2k^2}\right)du.$$

Hence noticing that

$$\bar{D}_{km}(x) = e^{-x^k/2k}\left\{x^{km} - \frac{km(km-k+1)}{1.k}x^{km-k} + \frac{km(km-k)(km-k+1)(km-2k+1)}{1.2.k^2}x^{km-2k} - \dots\right\},$$

we get

$$e^{-x^k/2k}\left[\bar{D}_{km}(x) + \frac{km(km-k+1)}{2.1!k}\bar{D}_{km-k}(x) + \cdots\right] = \frac{(-1)^m k^{1-1/k}}{2^{m+1/k}\Gamma(1/k)}\int_0^\infty \bar{D}_{km}(x).\,{}_0F_1\left(\frac{1}{k}\;;\;-\frac{u^k x^k}{2k^2}\right)du. \tag{21}$$

We know that $\bar{D}_{km}(x)$, satisfies the differential equation

$$\frac{d^2}{dx^2}\bar{D}_{km}(x) - \frac{k-2}{x}\frac{d}{dx}\bar{D}_{km}(x) + \{(\tfrac{1}{2}+mk)x^{k-2} - \tfrac{1}{4}x^{2k-2}\}\bar{D}_{km}(x) = 0. \tag{22}$$

Let

$$z = e^{-x^k/2k}\bar{D}^{2}_{kr}(x) = \bar{E}_{kr}(x).$$

If we write $2^{1/k}x$ for x, the equation (22) reduces to

$$\frac{d^2}{dx^2}\bar{D}_{km}(2^{1/k}x) - \frac{k-2}{x}\frac{d}{dx}\bar{D}_{km}(2^{1/k}x) + \{(1+2km)x^{k-2} - x^{2k-2}\}\bar{D}_{km}(2^{1/k}x) = 0.$$

The equation satisfied by $\bar{E}_{kr}(x)$ is

$$\frac{d^2z}{dx^2} + \left(x^{k-1} - \frac{k-2}{x}\right)\frac{dz}{dx} + (kr+1)x^{k-2}z = 0. \tag{23}$$

Let us assume that

$$\bar{D}_{km}(2^{1/k}x) = \sum A_{kr}\bar{E}_{kr}(x) = \sum A_{kr}e^{-x^k/2k}\bar{D}_{kr}(x).$$

On substituting in (22) and taking into account the differential equation and the recurrence relation satisfied by $\bar{E}_{kr}(x)$, *viz.*, equation (23) and

$$\bar{E}_{kr}(x)+x^{k-1}\bar{E}_{kr}(x)-\frac{kr}{x}\left(\bar{E}_{kr}(x)+(kr-k+1)\bar{E}_{kr-k}(x)\right)=0,$$

we get after a bit of reduction that

$$\sum A_{kr}\{(2mk-2rk)\bar{E}_{kr}(x)-kr(kr-k+1)\bar{E}_{kr-k}(x)\}=0,$$

whence

$$A_{kr-k}=A_{kr}.\frac{kr(kr-k+1)}{2(mk-rk+k)}.$$

Hence

$$A_{km-k}=A_{km}.\frac{km(km-k+1)}{1!\;2k}$$

$$A_{km-2k}=A_{km}.\frac{km(km-k)(km-k+1)(km-2k+1)}{2!\;2^2k^2}$$

and so on. Therefore

$$\bar{D}_{km}(2^{1/k}x)=A_{km}.e^{-x^k/2k}\{\bar{D}_{km}(x)+\frac{km(km-k+1)}{1!\;2k}\bar{D}_{km-k}(x)+\dots\}.$$

Equating the coefficient of x^{km}, we find that

$$A_{km}=2^m.$$

Hence the left side of (21) is

$$\frac{1}{2^m}\bar{D}_{km}(2^{1/k}x).$$

We therefore have

$$\bar{D}_{km}(2^{1/k}x)=\frac{(-1)^m k^{1-1/k}}{2^{1/k}\Gamma(1/k)}\int_0^\infty \bar{D}_{km}(u).{}_0F_1\left(\frac{1}{k}\;;\;-\frac{u^k x^k}{2k^2}\right)du. \tag{24}$$

If we put $k=2$, we arrive at the integral equation of Milne (1914).

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AN EXTENSION OF HAMY'S THEOREM TO ROTATING GASEOUS BODIES

By

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Introduction

M. Hamy (1889) proved the Theorem that "a heterogeneous mass of liquid rotating steadily about a fixed axis with a constant angular velocity cannot have an ellipsoidal density-distribution." The present author (Ghosh, 1947) in a previous paper has proved the impossibility of density-distribution in similar ellipsoids in a direct and simple way which shows that the result is true not only for the field due to self-gravitation of the rotating mass but also under any additional potential fields due to external sources such that Poisson's equation due to the density-distribution may remain unaffected.

This paper is an extension of the same theorem to the case of compressible fluid-masses, where the pressure is a function of the density alone, rotating under the same conditions, with angular velocities not necessarily constant. It proves that a heterogeneous mass of gas in which the pressure is a function of the density, rotating under its own gravitation cannot attain a similar-ellipsoidal density-distribution.

1. The equations of motion in Cartesian co-ordinates are

$$\omega^2 x = -\frac{\partial \Phi}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} \tag{1.1}$$

$$\omega^2 y = -\frac{\partial \Phi}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial y} \tag{1.2}$$

$$0 = -\frac{\partial \Phi}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z} \tag{1.3}$$

where ω, Φ, ρ, p denote the angular velocity (not uniform), the potential, the density and the pressure respectively and Φ satisfies at every point inside,

$$\nabla^2 \Phi = -4\pi G \rho \tag{1.4}$$

with

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

For any gas where p is a function of ρ alone, let

$$\int \frac{dp}{\rho} = f(\rho). \tag{1.5}$$

Hence, from (1.1), (1.2), (1.3),

$$\omega^2(xdx+ydy) = -d\Phi+\frac{dp}{\rho}. \tag{1.6}$$

and thus $\omega^2(xdx+ydy)$ must be a perfect differential, which proves that ω must be a function of x^2+y^2 only (Poincare's Theorem). From (1.3), (1.4) and (1.5), we have

$$\nabla^2\frac{\partial}{\partial z}f(\rho)+4\pi G.\frac{\partial\rho}{\partial z} = 0^*. \tag{1.7}$$

Now, if possible, let

$$\rho = \psi(\xi) \tag{1.8}$$

where

$$\xi = \alpha x^2+\beta y^2+\gamma z^2, \qquad 0\leq\xi\leq 1. \tag{1.9}$$

Then,

$$f(\rho) = f\{\psi(\xi)\} = \chi(\xi), \text{ say.} \tag{1.10}$$

Hence, from (1.7), we have

$$2(\alpha^2x^2+\beta^2y^2+\gamma^2z^2)\chi'''(\xi)+(\alpha+\beta+3\gamma)\chi''(\xi)+2\pi G\psi'(\xi) = 0. \tag{1.11}$$

Equation (1.11) is obviously integrable when, either

$$\alpha=\beta=\gamma, \quad \text{or} \quad \chi'''(\xi) = 0.$$

In the former case the density-distribution becomes spherical which contradicts our hypothesis. Hence for ellipsoidal distribution we must have,

$$\chi'''(\xi) = 0. \tag{1.12}$$

Integrating,

$$\chi''(\xi) = L, \tag{1.13}$$

where L is a constant ($\neq 0$). For, L can never vanish. Since in that case $\psi'(\xi)$ must also vanish and hence ρ will become a constant. From (1.11) and (1.13)

$$\psi'(\xi) = -\frac{\alpha+\beta+3\gamma}{2\pi G}.L,$$

whence,

$$\psi(\xi) = \rho = -\frac{(\alpha+\beta+3\gamma)}{2\pi G}.L\xi+\rho_0, \tag{1.14}$$

where,

$$\rho=\rho_0, \text{ when } x=y=z=\xi=0. \tag{1.14a}$$

Thus

$$\rho_0-\rho = \frac{L(\alpha+\beta+3\gamma)}{2\pi G}\xi. \tag{1.15}$$

For pressure, we have from (1.13)

$$\chi'(\xi) = L\xi+M \tag{1.16}$$

where M is a constant, and hence from (1.5) and (1.10)

$$\int\frac{dp}{\rho} = \chi(\xi) = \tfrac{1}{2}L\xi^2+M\xi+N \tag{1.16a}$$

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[*] The author desires to point out here that equation (1.7) plays a basic part in the study of all rotating gaseous models. It is intended to discuss this point elaborately in a subsequent paper.

where N is another constant. Hence, with (1.15) we have

$$\frac{dp}{d\xi} = (L\xi + M)\left\{\rho_0 - \frac{\alpha+\beta+3\gamma}{2\pi G}\cdot L\xi\right\} \tag{1.17}$$

leading to

$$p = -\frac{L^2(\alpha+\beta+3\gamma)}{6\pi G}\xi^3 + \frac{L}{2}\left\{\rho_0 - \frac{M(\alpha+\beta+3\gamma)}{2\pi G}\right\}\xi^2 + M\rho_0\xi + p_0, \tag{1.18}$$

where $p = p_0$ at $\xi = 0$.

Boundary Conditions. Stability requires that the density of the model must diminish outwards. Hence from (1.15) we must have

$$L > 0.$$

Again, as the boundary of the gaseous body is supposed to be free, p and ρ must vanish together on the surface. Hence in (1.15) ρ vanishes at $\xi = 1$, assuming the free-surface to be $\alpha x^2 + \beta y^2 + \gamma z^2 = 1$. And hence

$$L = \frac{2\pi\rho_0 G}{(\alpha+\beta+3\gamma)}. \tag{1.19}$$

With this value for L, (1.15) becomes

$$\rho = \rho_0(1-\xi). \tag{1.20}$$

The above value of ρ obviously satisfies the further condition that ρ must be positive everywhere inside the boundary. This follows at once from the inequality shown in (1.9).

With (1.19) and (1.20), (1.18) reduces, after simplification, to

$$p = \frac{L}{6\rho_0^2}\rho^3 - \frac{L+M}{2\rho_0}\rho^2 + \left(\frac{L+3M}{6}\rho_0 + p_0\right). \tag{1.21}$$

$M \geq 0$: As p_0, ρ_0 and L are all positive, it is obvious from (1.21) that p and ρ cannot vanish together if M is positive or zero. Hence there cannot be any equilibrium of such a type with a surface strictly free. Besides, equation (1.21) in ρ, for $p = 0$, does not give any positive root. Hence we conclude that when $M \geq 0$ there cannot exist any possible solutions with an ellipsoidal stratification. We shall next examine the case when M is negative.

From (1.21) we see that p will vanish on the surface where $\rho = 0$, only if,

$$M = -\frac{6p_0 + L\rho_0}{3\rho_0}. \tag{1.22}$$

But from (1.16) it is clear that M represents the value of $dp/\rho d\xi$ at the centre. That is

$$M = \left(\frac{1}{\rho}\frac{dp}{d\xi}\right)_{\xi=0}. \tag{1.23}$$

Thus it is possible for the pressure to vanish on the boundary given by $\rho = 0$ only when the value of $dp/d\xi$ ($\xi \equiv \alpha x^2 + \beta y^2 + \gamma z^2$) becomes negative at the centre. This does not vitiate the physical condition that the pressure attains a maximum at the centre.

From (1.18) by (1.19) wo have,

$$p = -\tfrac{1}{3}L\rho_0\xi^3 + \tfrac{1}{2}\rho_0(L-M)\xi^2 + M\rho_0\xi + p_0. \tag{1.24}$$

Hence,

$$dp/d\xi = (1-\xi)\{\rho_0 L\xi + M\rho_0\} = (1-\xi)\{\rho_0 L\xi - \tfrac{1}{3}(6p_0 + L\rho_0)\},$$

by (1.22).

As $1-\xi \geq 0$, $dp/d\xi$ will remain negative throughout if

$$p_0 \geq \tfrac{1}{3}\rho_0 L. \tag{1.25}$$

Thus under (1.25) the pressure will continuously diminish outwards and ultimately vanish at the surface, a necessary condition for any stable form of equilibrium. With (1.22), (1.24) becomes

$$p = (1-\xi)^2[p_0 - \tfrac{1}{3}\rho_0 L\xi] \tag{1.26}$$

As p must be positive everywhere inside it is enough to take

$$p_0 \geq \tfrac{1}{3}\rho_0 L \tag{1.26a}$$

as $\xi_{\max} = 1$.

Equations (1.25) and (1.26a) show that the same value of p_0 will make 'p' positive throughout the interior as will make it diminish steadily outwards.

It will be useful, here, to summarise the results that we have obtained so far, before proceeding any further.

We have found that Hamy's Theorem holds good in all cases except, perhaps, when the pressure and the density are given by (1.26) and (1.20) respectively and either of the conditions in (1.26a) holds. This leads us to *two likely exceptions* to Hamy's theorem according as p_0 equals or exceeds the value of $\tfrac{1}{3}\rho_0 L$.

We shall prove that in both cases Hamy's theorem does hold good if the field is due purely to self-gravitation.

Now we shall show that in every possible case of steady rotation about a fixed axis, as has been assumed at the start, the distribution of density must necessarily be spheroidal.

From (1.6) we have

$$\int \omega^2 r.dr = -\Phi + \int \frac{dp}{\rho} \tag{1.27}$$

where $r^2 = x^2 + y^2$.

Hence from (1.16*a*), the right hand side reduces to

$$-\Phi + \tfrac{1}{2}L\xi^2 + M\xi + N$$

where $L(>0)$, M and N are constants and $\xi \equiv \alpha x^2 + \beta y^2 + \gamma z^2$. Operating on both sides with ∇^2 and remembering that ω is a function of r alone, we have

$$\frac{1}{r}\frac{d}{dr}(\omega^2 r^2) = \{4\pi\rho_0 G + 2M(\alpha+\beta+\gamma)\} - 4L\{\alpha(\gamma-\alpha)x^2 + \beta(\gamma-\beta)y^2\}. \tag{1.28}$$

Equation (1.28) becomes integrable only when

$$\alpha = \beta. \tag{1.28a}$$

Thus the study of ellipsoidal distributions necessarily reduces to the study of spheroidal distributions in cases of steady motions about a fixed axis whatever be the angular velocity. Hence in further procedure it should be enough to consider ξ as $\alpha(x^2+y^2)+\gamma z^2$ or $\alpha r^2+\gamma z^2$.

Calculations for ω *and* Φ. From (1.28) we have, by integration, with (1.28a),

$$\omega^2 r^2 = \{2\pi\rho_0 G + M(\alpha+\beta+\gamma)\}r^2 - L\alpha(\gamma-\alpha)r^4 \tag{1.29}$$

the constant of integration being made to vanish as ω must remain finite on the axis.

The value of ω^2 given by (1.29), *viz.*,

$$\omega^2 = 2\pi\rho_0 G + M(\alpha+\beta+\gamma) - L\alpha(\gamma-\alpha)r^2 \tag{1.30}$$

holds good in both the exceptional cases mentioned before provided the appropriate values of L and M are substituted in (1.30).

Again from (1.29) we have

$$\int \omega^2 r\,dr = \tfrac{1}{2}\{2\pi\rho_0 G + M(\alpha+\beta+\gamma)\}r^2 - \tfrac{1}{4}L\alpha(\gamma-\alpha)r^4 + \text{const.} \tag{1.31}$$

Hence, from (1.27),

$$\Phi = \int \frac{dp}{\rho} - \tfrac{1}{2}\{2\pi\rho_0 G + M(\alpha+\beta+\gamma)\}r^2 + \tfrac{1}{4}L\alpha(\gamma-\alpha)r^4 + C, \tag{1.32}$$

where C is a const.

We have seen before that the possible exceptions to Hamy's Theorem are the cases where the density and pressure are given by (1.20) and (1.26) and either of the conditions in (1.26a) holds. That is, we have

$$\rho = \rho_0(1-\xi)\,;\ p = (1-\xi)^2[p_0 - \tfrac{1}{3}\rho_0 L\xi]\,;\ \text{and}\ p_0 = \tfrac{1}{3}\rho_0 L\ \text{or}\ p_0 > \tfrac{1}{3}\rho_0 L\,;$$

with

$$L = \frac{2\pi\rho_0 G}{\alpha+\beta+3\gamma}.$$

When $p_0 = \frac{1}{3}\rho_0 L$, we shall call the case, case (*a*); when $p_0 > \frac{1}{3}\rho_0 L$, we shall call the case, case (*b*). We shall discuss the two cases succesively in the next two articles.

2

Under the conditions for axial symmetry enforced by (1.28a) the equations of motion written in cylindrical co-ordinates take the form

$$\left.\begin{aligned} \omega^2 r &= -\frac{\partial\Phi}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} \\ 0 &= -\frac{\partial\Phi}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z} \end{aligned}\right\} \tag{2.1}$$

with,

$$\nabla^2\Phi \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{\partial^2\Phi}{\partial z^2} = -4\pi G\rho. \tag{2.2}$$

We have, further, p and ρ given by (1.26) and (1.20) together with

$$p_0 = \tfrac{1}{3}\rho_0 L. \tag{2.3}$$

With this, (1.26) gives

$$p = p_0(1-\xi)^3 \tag{2.4}$$

and, of course,

$$\rho = \rho_0(1-\xi), \tag{2.5}$$

as before, and

$$\xi = \alpha r^2 + \gamma z^2. \tag{2.6}$$

Besides, from (1.22), with (2.3), we have

$$M = -L \tag{2.7}$$

and from (1.19).

$$L = \frac{2\pi\rho_0 G}{2\alpha+3\gamma}. \tag{2.8}$$

Hence from (2.3),

$$p_0 = \frac{2\pi\rho_0{}^2 G}{3(2\alpha+3\gamma)}. \tag{2.9}$$

From (2.4) and (2.5), the (p, ρ) relation is given by.

$$p = k\rho^3 \tag{2.10}$$

where

$$k = p_0/\rho_0{}^3. \tag{2.11}$$

The free-surface or the boundary, given by $p = \rho = 0$, is

$$\alpha r^2 + \gamma z^2 = 1 \tag{2.12}$$

and for oblateness

$$\gamma > \alpha. \tag{2.13}$$

In this case, (1 30) gives

$$\omega^2 = 3k\rho_0{}^2\{2\gamma - \alpha(\gamma-\alpha)r^2\}. \tag{2.14}$$

This shows that ω remains real throughout the mass. Again (1.32) considered with (1.16a) reduces in this case to

$$\Phi = \tfrac{3}{2}k\rho_0{}^2(1-\alpha r^2-\gamma z^2)^2 - \tfrac{3}{4}k\rho_0{}^2 r^2[4\gamma - \alpha(\gamma-\alpha)r^2] + C, \tag{2.15}$$

where C is a constant.

The function Φ given by (2.15) satisfies the equations (2.1) and (2.2), is finite and continuous at every point inside and its first and second derivatives satisfy the necessities of a potential function inside a distribution. The question that now remains to be settled is whether this Φ represents the Newtonian gravitational potential due to the distribution ρ or not. We shall designate the Newtonian potential due to self-gravitation of the mass by V.

We can easily convince ourselves that the potential V due to the mass distribution (2.5) cannot be the same function as Φ. If for the moment we assume that Φ and V are the same function then their values should agree at all points within the mass where (2.15) holds. V can be calculated by simple integration at two points, *viz.*, at the centre and at the poles of the spheroidal mass. It can be shown that Φ and V do not agree simultaneously at both these points. The identification of Φ and V at the centre determines the value of the unknown constant C in (2.15). This enables us to calculate the value of Φ exactly at the poles, The co-ordinates of the centre are (0, 0) and those of the upper pole (which we shall denote as B) are $(0, \gamma^{-\frac{1}{2}})$. We find after some calculations that

$$V(0,\ 0) = \frac{\pi\rho_0 G}{\{\alpha(\gamma-\alpha)\}^{\frac{1}{2}}} \cdot \tan^{-1}\left(\frac{\gamma-\alpha}{\alpha}\right)^{\frac{1}{2}} \tag{2.16}$$

and

$$V_B = \frac{\pi\rho_0 G}{8\{\alpha(\gamma-\sigma)\}^{\frac{1}{2}}} \cdot \left(\frac{\gamma}{\gamma-\alpha}\right)^2 \left[3 \tan^{-1}\left(\frac{\gamma-\alpha}{\alpha}\right)^{\frac{1}{2}} - \frac{\{\alpha(\gamma-\alpha)\}^{\frac{1}{2}}(5\gamma-2\alpha)}{\gamma^2}\right]. \tag{2.17}$$

From (2.15), we obtain

$$\Phi(0,\ 0) = \tfrac{3}{8}k\rho_0{}^2 + C.$$

Hence, equating $\Phi(0, 0)$ to $V(0, 0)$, we have,

$$C = -\frac{\pi\rho_0 G}{2\alpha+3\gamma} + \frac{\pi\rho_0 G}{\{\alpha(\gamma-\alpha)\}^{\frac{1}{2}}} \cdot \tan^{-1}\left(\frac{\gamma-\alpha}{\alpha}\right)^{\frac{1}{2}} \tag{2.18}$$

by (2.9) and (2.11). Again, from (2.15),

$$\Phi_B = C. \tag{2.19}$$

Hence equating Φ and V at B, we must have

$$-\frac{\pi\rho_0 G}{2\alpha+3\gamma} + \frac{\pi\rho_0 G}{\{\alpha(\gamma-\alpha)\}^{\frac{1}{2}}} \cdot \tan^{-1}\left(\frac{\gamma-\alpha}{\alpha}\right)^{\frac{1}{2}}$$

$$= \frac{\pi\rho_0 G}{3\{\alpha(\gamma-\alpha)\}^{\frac{1}{2}}} \cdot \left(\frac{\gamma}{\gamma-\alpha}\right)^3 \left[3 \tan^{-1}\left(\frac{\gamma-\alpha}{\alpha}\right)^{\frac{1}{2}} - \frac{\{\alpha(\gamma-\alpha)\}^{\frac{1}{2}}(5\gamma-2\alpha)}{\gamma^3}\right]. \tag{2.20}$$

Now if a, c be the semi-axes of the elliptic section through the axis of rotation, we have

$$\alpha = \frac{1}{a^2}; \quad \gamma = \frac{1}{c^2}; \quad \frac{\alpha}{\gamma} = \frac{c^2}{a^2} = 1-e^2; \text{ and } \frac{\gamma-\alpha}{\alpha} = \frac{e^3}{1-e^3}$$

where e is the eccentricity and hence must be less than 1.

Expressing equation (2.20) in terms of e we see that the identity of the two sides does not hold for any positive value of e less than 1. Hence we conclude that Φ and V do not agree, at the point B in any real case. The case for exact sphericity, *i.e.*, when $e = 0$, is discussed independently, because the expression for V at any point inside can be obtained easily.

It may be remarked that the limiting case when $e = 0$, is physically important and interesting since steller bodies inspite of their small rotations exhibit practically spherical forms.

The spherical case ($e = 0$): The results for this case are obviously obtained from the previous one by putting $\gamma = \alpha$, and hence we have,

$$\left.\begin{aligned} \rho &= \rho_0(1-\alpha R^2), \\ p &= p_0(1-\sigma R^2)^3 \end{aligned}\right\} \tag{2.21}$$

where $R^2 = r^2 + z^2$, $p_0 = k\rho_0{}^3$, $\rho_0\alpha = 2\pi G/15k$.

(2.14) gives

$$\omega^2 = \frac{4\pi\rho_0 G}{15} \tag{2.22}$$

showing that ω must be a constant, and lastly, Φ takes the form

$$\tfrac{3}{2}k\rho_0{}^2(1-\alpha R^2)^2 - 3k\rho_0{}^2\alpha r^2 + \lim_{\gamma\to a} C.$$

But

$$\lim_{\gamma\to a} C = \frac{4\pi\rho_0 G}{5\alpha}.$$

Hence we have, putting $\alpha = 1/a^2$,

$$\frac{\Phi}{\pi\rho_0 G} = a^2 - \frac{2}{5}R^2 + \frac{1}{5}\,\frac{R^4}{a^2} - \frac{2}{5}r^2. \tag{2.23}$$

Now, in this case, we obtain easily,

$$\frac{V}{\pi\rho_0 G} = a^2 - \frac{2}{3}R^2 + \frac{1}{5}\,\frac{R^4}{a^2}. \tag{2.24}$$

Equations (2.23) and (2.24) show that Φ and V though identical at the centre are not so at every other point. In fact, we have,

$$\frac{\Phi - V}{\pi\rho_0 G} = \frac{2}{15}(2z^2 - r^2) = V', \text{ say.} \tag{2.25}$$

We can easily verify that

(i) $\nabla^2 V' = 0$ at every point inside;
(ii) $\nabla V' = 0$ at $r = 0, \; z = 0$;
(iii) $V' = 0$ at $r = 0, \; z = 0$;
(iv) $V' \neq 0$ at $r = 0, \; z = a$.

The discrepancy in Φ and V as shown by (2.25) leads to the conclusion that equilibrium of the type discussed is not possible even in the limiting case when the density distribution becomes spherical, provided the potential field is due purely to self-gravitation.

On the other hand, if we suppose that the density distribution in (2.21) is maintained under the additional influence of external gravitating masses, the potential due to such additional masses should be given by (2.25) as $\pi\rho_0 G V'$. But as V' vanishes and changes sign it cannot be due to any distribution of real matter. Hence spherical distribution of density of the nature (2.21) is not possible under any gravitational field whatsoever.

3

Case (*b*). $\rho_0 > \frac{1}{3}\rho_0 L$:

In this case the equations (2.1), (2.2), (2.5), (2.6) and (2.8) hold good. The pressure p is given by (1.26) and for M we have the value given by (1.22). In addition we have $\rho_0 > \frac{1}{3}\rho_0 L$, and hence we put

$$p_0 = \tfrac{1}{3}\rho_0 L(1+\mu^2) \tag{3.2}$$

where

$$\mu^2 > 0. \tag{5.2}$$

[$\mu = 0$, gives us the case discussed in the previous article, and hence case (*b*) can be regarded as more general than case (*a*)].

From (1.26) by (3.1), we have

$$p = \tfrac{1}{3}\rho_0 L[(1-\xi)^2 + \mu^2(1-\xi)^2]. \tag{3.3}$$

But $\rho = \rho_0(1-\xi)$, hence

$$\int \frac{dp}{\rho} = \tfrac{1}{6}L[3(1-\xi)^2 - 4\mu^2\xi]. \tag{3.4}$$

Again, from (1.22) and (3.1) we have

$$M = -L(1+\tfrac{2}{3}\mu^2). \tag{3.5}$$

With this value for M, (1.32) becomes

$$\Phi = \tfrac{1}{6}L[3(1-\xi)^2 - 4\mu^2\xi] - \tfrac{1}{2}L[\{2\gamma - \tfrac{2}{3}\mu^2(2\alpha+\gamma)\}r^2 - \tfrac{1}{2}\alpha(\gamma-\alpha)r^4] + C \tag{3.6}$$

where C is a constant. Equation (3.6) can also be written as

$$\Phi = \tfrac{1}{2}L[(1-\xi)^2 - r^2\{2\gamma - \tfrac{1}{2}\alpha(\gamma-\alpha)r^2\} + \tfrac{2}{3}\mu^2\gamma(r^2-2z^2)] + C. \tag{3.7}$$

As in the case of Art. 2, the question that now presents itself is whether the function Φ that appears in (3.6) or (3.7) represents for any real value of μ, the actual gravitational potential (V) due to the density distribution $\rho_0(1-\xi)$, confined within the boundary $\alpha r^2 + \gamma z^2 = 1$.

We shall prove that Φ cannot represent the Newtonian potential (V) due to self-gravitation except in the limiting case when $\gamma \to \alpha$ which however becomes trivial as the angular velocity for that case vanishes everywhere.

The limiting case $\gamma \to \alpha$. We propose first to examine if there is a real value for μ which will make Φ agree with V in the limiting case where $\gamma \to \alpha$, *i.e.*, when the distribution becomes spherical. We find that $\mu = 1$ satisfies the requirement but the corresponding ω vanishes, as has already been mentioned. From (3.7),

$$\lim_{\gamma\to\alpha} \Phi = \tfrac{1}{2}L\left[\left(1-\frac{R^2}{a^2}\right)^2 - \frac{2r^2}{a^2}(1-\tfrac{2}{3}\mu^2) - \frac{4}{3}\frac{\mu^2 z^2}{a^2}\right] + C' \tag{3.8}$$

where

$$R^2 = r^2 + z^2; \quad \alpha = \gamma = \frac{1}{a^2}; \quad C' = \lim_{\gamma\to\alpha} C.$$

With $\mu = 1$, (3.8) reduces to

$$\Phi = \pi\rho_0 G\left[\frac{a^2}{5} - \frac{2}{3}R^2 + \frac{1}{5}\frac{R^4}{a^2}\right] + C', \tag{3.9}$$

when we put

$$L = \tfrac{2}{5}\pi\rho_0 G a^2, \tag{3.10}$$

to which (2.8) is reduced in the limiting case.

But the density distribution and the boundary being the same as in the limiting case of Art. 2, we have V given by (2.24) and hence

$$V = \pi\rho_0 G\left[a^2 - \frac{2}{3}R^2 + \frac{1}{5}\frac{R^4}{a^2}\right]. \tag{3.11}$$

Hence Φ and V will agree exactly everywhere inside the mass if

$$C' = \tfrac{4}{5}\pi\rho_0 G.a^2; \quad \mu = 1. \tag{3.12}$$

Now we find from (1.30)

$$\omega^2 = 2\pi\rho_0 G + M(2\alpha+\gamma)$$

But

$$M = -L(1+\tfrac{2}{3}\mu^2) = -\tfrac{2}{3}\pi\rho_0 Ga^2. \tag{3.13}$$

Hence

$$\omega = 0. \tag{3.14}$$

Hence the limiting case of case (*b*) reduces to that of a non-rotating model when equilibrium is maintained purely under self-gravitation, and as such the result becomes trivial.

The general case. To discuss the general case we go back to equation (3.7). Substituting for L and ξ from (2.6) and (2.8), we have

$$\frac{2\Phi}{\pi\rho_0 a^2 c} = \frac{2}{2c^2+3a^2}\left[c - 2r^2.\frac{3c^2+a^2(3-\mu^2)}{3a^2c} - 2z^2.\frac{3+2\mu^3}{3c}\right.$$
$$\left. + \frac{2r^2z^3}{a^2c} + r^4.\frac{a^2+c^2}{2a^4c} + \frac{z^4}{c^3}\right] + C', \tag{3.15}$$

where $\alpha = 1/a^2$, $\gamma = 1/c^2$ and C' is a constant.

The Newtonian gravitational potential V for the distribution $\rho = \rho_0\,(1-r^2/a^2-z^2/c^2)$, however, can be obtained by a formula given by Hobson (1896). We obtain therefrom

$$V = \frac{\pi\rho_0 a^2 c}{2}\int_0^\infty \frac{d\theta}{(a^2+\theta)(c^2+\theta)^{\frac{1}{2}}}\left\{1 - \frac{r^2}{(a^2+\theta)} - \frac{z^2}{(c^2+\theta)}\right\}^2.$$

Hence,

$$\frac{2V}{\pi\rho_0 a^2 c} = A_0 - 2A_1 r^2 - 2A_2 z^2 + 2A_3 r^2 z^2 + A_4 r^4 + A_5 z^4 \tag{3.16}$$

where the coefficients A_0, A_1, A_2, A_3, A_4, A_5, are given by

$$\left.\begin{aligned} A_0 &= \int_0^\infty \frac{d\theta}{(a^2+\theta)(c^2+\theta)^{\frac{1}{2}}}; \quad A_1 = \int_0^\infty \frac{d\theta}{(a^2+\theta)^2(c^2+\theta)^{\frac{1}{2}}}; \quad A_2 = \int_0^\infty \frac{d\theta}{(a^2+\theta)(c^2+\theta)^{3/2}}; \\ A_3 &= \int_0^\infty \frac{d\theta}{(a^2+\theta)^2(c^2+\theta)^{3/2}}; \quad A_4 = \int_0^\infty \frac{d\theta}{(a^2+\theta)^3(c^2+\theta)^{\frac{1}{2}}}; \quad A_5 = \int_0^\infty \frac{d\theta}{(a^2+\theta)(c^3+\theta)^{5/2}} \end{aligned}\right\}. \tag{3.17}$$

Hence for the identity of Φ and V we must have,

$$\left.\begin{array}{lll} A_0 = \dfrac{2c}{2c^2+3a^2}+C'; & A_1 = \dfrac{2}{3}\,\dfrac{3c^2+a^2(3-\mu^2)}{(2c^2+3a^2)a^2c}; & A_2 = \dfrac{2}{3}\,\dfrac{3+2\mu^2}{c(2c^2+3a^2)}; \\ A_3 = \dfrac{2}{a^2c(2c^2+3a^2)}; & A_4 = \dfrac{a^2+c^2}{(2c^2+3a^2)a^4c}; & A_5 = \dfrac{2}{c^3(2c^2+3a^2)}. \end{array}\right\} \qquad (3.18)$$

It is found that the last three equations in (3.18) become inconsistent when their values from (3.17) are substituted unless $a = c$. That means Φ and V cannot be identical unless the spheroid reduces to a sphere. Hence we conclude that a strictly spheroidal distribution of density is impossible in compressible fluid bodies rotating under self-gravitation.

This proves conclusively that Hamy's theorem (for distribution in similar ellipsoid) can be extended without exception to the case of non-homogenous gaseous masses, with pressure as a function of the density only, and rotating with variable angular velocities in general.

In conclusion, I take this opportunity to offer my grateful thanks to Prof. N. R. Sen for his kind interest and encouragement in the preparation of this paper.

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NOTE ON A CLASS OF INFINITE RIEMANNIAN INTEGRALS

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Abstract

The present paper deals with several categories of infinite (Riemannian) integrals of a real function, involving several positive parameters. The sufficient conditions of uniform and absolute convergence of any such integral have been touched upon, and its functional value (if and when convergent) in terms of the parameters has been scrutinised with special reference to its properties of continuity and differentiability. One of the remarkable categories of infinite integrals, discussed in this paper, is that in which the integrand, besides involving a number of positive parameters, contains, as a factor, one or more of the three functions $\pi(t)$, $\theta(t)$ and $\psi(t)$, so conspicuous in the Modern Theory of Numbers (particularly, Theory of Prime Numbers). For this purpose free use has been made of the (known) "asymptotic values" of the three functions. The paper concludes with a laconic reference to some of the more common infinite integrals, which play an important role in the Theory of Attraction and Potential and Hydrodynamics.

A good deal of the results embodied in this paper is believed to be original.

Introduction

As mentioned in the Abstract, the main purpose of the present investigation is:

(*i*) to study the conditions of uniform and absolute convergence of an infinite integral I of a real function, involving a number of positive parameters;

(*ii*) to characterise the functional value of I in terms of the parameters; and

(*iii*) to establish the legitimacy of differentiation under the integral sign (in *favourable* circumstances).

The integrands of certain varieties of infinite integrals, discussed in this paper, contain the three familiar functions $\pi(t)$, $\theta(t)$ and $\psi(t)$ in the "Theory of (Prime) Numbers." As is well-known these functions are "almost continuous" and therefore integrable over any *bounded* domain, and indirectly give rise to certain subsidiary functions, which are integrable over any domain, *bounded* or *unbounded*. The "subsidiary" functions are a component part of the integrands of some of the infinite integrals, disposed of in this paper.

The paper concludes with an enumeration—with brief annotations—of certain well-known (infinite) integrals, which figure so conspicuously in Applied Mathematics.

It may be pointed out once for all that the term "domain," used in this paper, stands for "a domain of positive quantities" and that the contraction *w.r.t.* means "with respect to."

The paper is believed to contain not a little original matter.

1. Consider an infinite integral I of the type

$$I \equiv \int_d^\infty V.dt, \quad (d \geq 0) \tag{1}$$

wherein the integrand V, regarded as a function of the real variable t, is given by:

$$V = \frac{f(t)}{W}, \tag{2}$$

it being postulated:

(i) that $f(t)$ is a polynomial of degree m,

(ii) that $W = \prod_{r=1}^{r=n} (a_r + t)^{\alpha_r}$,

(iii) that $a_r > 0$, $\alpha_r > 0$ for *every* r, and

(iv) that the constants $\{a_r\}$ and $\{\alpha_r\}$—be they integral or fractional—conform to the relation:

$$q \equiv p - m > 1, \quad \text{where} \quad p \equiv \sum_{r=1}^{r=n} \alpha_r. \tag{3}$$

Now because $a_r + t \geq t$ for *every* r, we must have

$$W = \prod_{r=1}^{r=n} (a_r + t)^{\alpha_r} > t^{\sum_{r=1}^{r=n} \alpha_r}, \text{ i.e., } > t^p,$$

so that, when t is very large,

$$|V| = \frac{|f(t)|}{|W|} < C.\frac{t^m}{t^p},$$

(where C is a *finite* constant), *i.e.*,

$$|V| = O(1/t^q).$$

If, then, an "auxiliary" infinite integral J—which is of course absolutely convergent—be defined by:

$$J = \int_d^\infty \frac{dt}{t^q}, \quad (d > 0,\ q > 1),$$

we observe that the quotient of the integrand of I, divided by that of J, *viz.*,

$$\frac{V}{1/t^q}$$

has its modulus *always bounded*, whatever positive value (finite or infinite) be ascribed to the variable t. So by the analogue of Weirstrass's **M**-test or by the generalised form of

De la Vallée Poussin's test,† the integral I is absolutely and uniformly convergent for *all finite* (real) positive values of $(a_1, a_2, \ldots, a_n)$. Further, V being a continuous function of the a-parameters (as also of t), we infer, on the strength of a well-known lemma on Analysis, that I is a *continuous* function of $(a_1, a_2, \ldots, a_n)$.

To scrutinise other aspects of this functionality we proceed as follows:

In the first place we remark that the integrand V (of I) is not only a continuous function of $(a_1, a_2, \ldots, a_n)$ but also differentiable any number of times *w.r.t.* each of them. In point of fact, the partial differential coefficient of V *w.r.t.* a_1, may be written as:

$$\left.\begin{aligned} \frac{\partial V}{\partial a_1} &= -\alpha_1 \frac{f(t)}{(a_1+t)W}; \\ \text{so that} \qquad & \\ \int_d^\infty \frac{\partial V}{\partial a_1} dt &= -\alpha_1 \int_d^\infty \frac{f(t)\, dt}{(a_1+t)^{\alpha_1+1}(a_2+t)^{\alpha_2}(a_3+t)^{\alpha_3} \ldots (a_n+t)^{\alpha_n}} \end{aligned}\right\} \tag{4}$$

The multiplier $(-\alpha_1)$ being ignored, we notice that the infinite integral on the R. H. S. of (4) is *almost identical* with I, the *only* difference being that the indices of the binomial factor (a_1+t) in the denominators of the two integrands are α_1 in one case and (α_1+1) in the other.

Since $(\alpha_1+1)+\alpha_2+\alpha_3+\cdots+\alpha_n-m = 1+(p-m) > 2$, we gather from what has been proved heretofore that, just like I, the infinite integral (4), *viz.*,

$$\int_d^\infty \frac{\partial V}{\partial a_1} dt$$

is also absolutely and uniformly convergent for *all* positive values of $(a_1, a_2, \ldots, a_n)$. Further in view of the fact that $\partial V/\partial a_1$ is a continuous function of a_1, we conclude that the original integral I must be differentiable at least once *w.r.t.* a_1 and that

$$\frac{\partial I}{\partial a_1} = \int_d^\infty \frac{\partial V}{\partial a_1} . dt. \tag{5}$$

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† The test in question may be enunciated in the following *convenient* form :—

If an auxiliary infinite integral J, *viz.*,

$$\int_d^\infty \chi(x))dx$$

be absolutely convergent and if the function $\psi(x, \alpha)$—supposed bounded—fulfils the condition that the modulus, *viz.*,

$$\left|\frac{\psi(x, \alpha)}{\chi(x)}\right|,$$

remains always *bounded*, as x ranges freely in the *unbounded* domain (d, ∞) and the (real) parameter α moves freely in a specified bounded domain (a', b'), then the infinite integral

$$\int_d^\infty \psi(x, \alpha)\, dx$$

will be absolutely and uniformly convergent for *all* values of α belonging to the interval (a', b')." Carslaw, 1930, §89; Whittaker and Watson, 1940, §4.431).

Considerations of symmetry shew that similar results must hold for differential coefficients *w.r.t.* other parameters, *i.e.*,

$$\frac{\partial I}{\partial a_m} = \int_d^\infty \frac{\partial V}{\partial a_m}.dt, \quad (m = 1, 2, \ldots, n). \tag{6}$$

Inasmuch as each of the n infinite integrals, comprised by (6), is ***uniformly convergent*** for all positive values of $(a_1, a_2, \ldots, a_n)$ and $\partial V/\partial a_m$ is a continuous function of t and $a_1, a_2, \ldots, a_n$ and is itself differentiable indefinitely *w.r.t.* each of them, it is crystal-clear that any of the n integrals, marked (6), must also admit of differentiation under the sign of integration *w.r.t.* each of the n parameters. In other words, the rule for partial differentiation *w.r.t.* $\{a_r\}$ under the sign of integration is valid in its most ***generalised*** aspect. We may accordingly summarise our final conclusions in the form of a regular proposition, *viz.*,

Proposition A. *If*

$$I \equiv \int_d^\infty V dt \equiv \int_d^\infty \frac{f(t)}{\prod\limits_{r=1}^{r=n}(a_r+t)^{\alpha_r}}\, dt. \tag{I}$$

where $f(t)$ is a polynomial of degree m and $d \geq 0$, $a_r > 0$ and $\alpha_r > 0$ (for every r) and $\sum\limits_{r=1}^{r=m} \alpha_r - m > 1$, then I must be absolutely and uniformly convergent for all finite values of $(a_1, a_2, \ldots, a_n)$ and its functional value in terms of those parameters must be ***not only continuous*** *but also differentiable indefinitely w.r.t. each of them. Moreover, the typical form of a mixed partial differential coefficient of I is given by:*

$$\Delta I = \int_d^\infty \Delta V.dt, \tag{II}$$

where Δ stands for the (mixed) partial differential operator:

$$\frac{\partial^{k_1+k_2+k_3+\ldots+k_n}}{\partial a_1^{k_1}\partial a_2^{k_2}\partial a_3^{k_3}\ldots\partial a_n^{k_n}}. \tag{III}$$

Furthermore, subject to the afore-mentioned restrictions, every derivative infinite integral of the type (II) ***is itself absolutely and uniformly convergent for all finite values af $(a_1, a_2, \ldots, a_n)$.***

2. A little reflection shows that Proposition A of the previous article remains *perfectly valid* even when the function $f(t)$ is no longer a polynomial of degree m but is rather an arbitrary function, which is continuous and differentiable indefinitely for *all* positive values of t, and at the same time fulfils the condition:

$$f(t) = O(t^m), \quad \text{where } t \text{ is very large.}$$

We shall in the succeeding articles deal with another category of infinite integrals, in which the integrand V (of I) is modified in various ways. As a preliminary to an adequate discussion of such integrals, it will be necessary in the next article to make a short digression on three remarkable functions of a real variable, so well-known in the Theory of Numbers (particularly, Theory of prime numbers).

3. Supposing t to be any given *positive* quantity (integral or fractional), we shall introduce three functions $\pi(t)$, $\theta(t)$ and $\psi(t)$ in the following manner:

$\pi(t) =$ the number of prime numbers $\leq t$;

$\theta(t) =$ the sum of the logarithms of all primes $\leq t$; *i.e.*, $\theta(t) = \sum\limits_{p<t} \log p$;

and $\psi(t) =$ the sum of the logarithms of all primes, whose *powers* are $\leq t$, *i.e.*,

$$\psi(t) = \sum_{p^m<t} \log p.$$

Then we have it, from the Theory of Numbers (Hardy and Wright, 1945), that, when t is very large,

$$\left.\begin{array}{l} \pi(t) \sim t/(\log t), \\ \text{and} \quad \theta(t) \sim \psi(t) \sim t. \end{array}\right\} \tag{1}$$

The very definition shows that, in any *bounded* domain (a, b), where $0 < a < b$, each of the three functions

$$\pi(t), \quad \theta(t), \quad \psi(t) \tag{2}$$

is bounded and "almost continuous"* and is therefore integrable over any part of the *bounded* domain. Next, the three subsidiary functions:

$$\frac{\pi(t) \log t}{t}, \quad \frac{\theta(t)}{t}, \quad \frac{\psi(t)}{t} \tag{3}$$

are bounded and "almost continuous" throughout any domain (*bounded or unbounded*); but although integrable over any *bounded* domain, none of the functions (3) is integrable over the *unbounded* domain (d, ∞). However it is easy to see that, by a slight modification of the denominator, each of the functions (3) can be rendered integrable over the *unbounded* domain (d, ∞). As a matter of fact, the absolute convergence of the infinite integral

$$\int_d^\infty \frac{dt}{t^\delta}, \; (d > 0, \; \delta > 1)$$

being kept in view, the absolute convergence of each of the infinite integrals:

$$\int_d^\infty \frac{\pi(t) \log t}{t^\beta} dt, \quad \int_d^\infty \frac{\theta(t)}{t^\beta} dt, \quad \int_d^\infty \frac{\psi(t)}{t^\beta} dt, \quad (d > 1, \; \beta > 2) \tag{4}$$

becomes manifest.

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* As usual, a function $f(t)$ of a real variable is said to be "*almost continuous*" in an interval, provided that it is generally continuous save as to a set of points of (ordinary) discontinuity of *zero measure*. In view of the fact that the (ordinary) discontinuities of $\pi(t)$ consist of the *enumerable* sequence of prime numbers 2, 3, 5, 7, . . ., it is manifest that $\pi(t)$ is "almost continuous" throughout any bounded domain. Similar remarks obviously apply to the functions $\theta(t)$ and $\psi(t)$.

The reader can easily multiply instances of absolutely convergent integrals of the above description. Thus, for example, the integral

$$\int_d^\infty \frac{\{\pi(t)\}^\lambda \{\theta(t)\}^\mu \{\psi(t)\}^\nu (\log\, t)^\lambda}{t^{\lambda+\mu+\nu+\lambda+\beta}}\, dt, \quad (\lambda > 0,\ \mu \geq 0,\ \nu \geq 0,\ d \geq 1,\ \beta \geq 1)$$

can be readily verified to be absolutely convergent.

4. We shall now take into consideration certain types of infinite integrals of a more complicated type than those discussed in the foregoing articles. For this purpose we shall use the symbol V in the sense of Art. 1, *as modified by Art. 2*; that is to say, the defining equation for V will be taken to be

$$V = \frac{f(t)}{W}, \tag{1}$$

where:

(*i*) $f(t)$ is a function (rational or transcendental), which is continuous and differentiable indefinitely for *all* positive values of the real variable t and moreover tallies with the condition:

$$f(t) = O(t^m), \quad \text{when } t \text{ is very large}, \tag{2}$$

(*ii*) $W = \prod_{r=1}^{r=n} (a_r + t)^{\alpha_r}$, where $a_r > 0$ and $\alpha_r > 0$ for *every* r,

and (*iii*) $q = p - m > 1$, where $p = \sum_{r=1}^{r=n} \alpha_r$.

The continuous function V being defined once for all in the afore-said manner, let us now introduce an infinite integral I as follows:

$$I \equiv \int_d^\infty VU dt, \quad (d \geq 0), \tag{3}$$

where the symbol U may stand for *any one* of the functions:

$$\left.\begin{array}{l} \dfrac{\pi(t)\log t}{t^\beta},\quad \dfrac{\theta(t)}{t^\beta},\quad \dfrac{\psi(t)}{t^\beta},\ (\beta > 2) \\ \text{and} \\ \dfrac{\{\pi(t)\}^\lambda \{\theta(t)\}^\mu \{\psi(t)\}^\nu (\log t)^\lambda}{t^{\lambda+\mu+\nu+\beta}} \end{array}\right\} \tag{4}$$

(where $\lambda > 0$, $\mu > 0$, $\nu > 0$, $d > 0$, $\beta > 1$).

Inasmuch as the *modus operandi* is essentially the same in all the cases, we may, for the sake of precision, concentrate our attention upon only one of them (say, the first). Thus we shall set

$$U = \frac{\pi(t)\log t}{t^\beta}, \quad (\beta > 2). \tag{5}$$

Now by (4) of Art. 3, the definite integral J, defined by

$$J \equiv \int_d^\infty U dt, \quad (d \geq 0,\ \beta \geq 2) \tag{6}$$

is absolutely convergent. If then we compare the integrands of I and J, as defined by (3) and (6), we find

$$\left|\frac{VU}{U}\right| = |V| = \frac{|f(t)|}{|W|} = O\left(\frac{t^m}{t^p}\right) = O\left(\frac{1}{t^q}\right)$$

(when t is very large) by (2).

Because $q > 1 > 0$, we learn that when t is very large,

$$\left|\frac{VU}{U}\right| \text{ is very small.}$$

Needless to say, so long as t is finite, $\left|\frac{VU}{U}\right|$ is also finite. In other words, the modulus of the quotient of the integrand of I, divided by that of J, is *bounded* for *all* positive values of t (*finite or infinite*). Since this is true for *all* positive values of the parameters $(a_1, a_2, \ldots, a_n)$, we conclude—in accordance with the analogue of Weirstrass's **M**-test or with the generalised test of De la Vallée Poussin (as mentioned in the foot-note to Art. 1)—that the integral I, as defined by (3), is *uniformly* (and absolutely) convergent for all positive values of the a-parameters.

Admittedly the integrand of (3), *viz.*, VU is *not* a continuous function of $(t, a_1, a_2, \ldots, a_n)$ although V itself is such; nevertheless the integral I is uniformly convergent and the function U, *though not continuous*, is a bounded integrable function of t. Hence appealing to Carslaw (1926, §84), we infer that I is a continuous function of $(a_1, a_2, \ldots, a_n)$.

Now remarking that U is quite independent of the parameters, we have:

$$\frac{\partial}{\partial a_r}(VU) = U\frac{\partial V}{\partial a_r}, \quad (r = 1, 2, 3, \ldots, n),$$

so that by (4) of Art. 1,

$$\int_d^\infty \frac{\partial}{\partial a_r}(VU)dt = -a_r\int_d^\infty \frac{f(t)Udt}{(a_1+t)^{a_1+1}(a_2+t)^{a_2}(a_3+t)^{a_3}\ldots(a_n+t)^{a_n}}.$$

Repetition of the previous line of argument makes it plain that the integral

$$\int_d^\infty \frac{\partial}{\partial a_r}(VU)dt \quad (r = 1, 2, 3, \ldots, n)$$

is uniformly convergent for all positive values of a_r.

Hence referring to §86 (Carslaw, 1926) we gather that the integral I is differentiable partially *w.r.t.* each of the a-parameters; the precise formula is

$$\frac{\partial I}{\partial a_r} = \int_d^\infty \frac{\partial}{\partial a_r}(VU)dt, \quad (r = 1, 2, 3, \ldots, n).$$

Palpably, this result is susceptible to generalisation in the following form:—

$$\Delta I = \int_d^\infty \Delta(VU).dt,$$

where Δ stands for the mixed differential operator:

$$\frac{\partial^{k_1+k_2+\cdots+k_n}}{\partial a_1^{k_1}\,\partial a_2^{k_2}\,\ldots\,\partial a_n^{k_n}}.$$

In other words, Proposition A of §1 holds *in toto*, when the integrand V is multiplied by the "almost continuous" function U, *viz.*

$$U \equiv \frac{\pi(t)\log t}{t^{\beta}}, \quad (\beta > 2).$$

By parallel reasoning one can easily satisfy oneself that Proposition A of §1 remains valid just as before, when the integrand V is multiplied by any one of the following functions, *viz.*

$$\frac{\theta(t)}{t^{\beta}}, \quad \frac{\psi(t)}{t^{\beta}}, \quad (\beta > 2)$$

and

$$\frac{\{\pi(t)\}^{\lambda}\,\{\theta(t)\}^{\mu}\,\{\psi(t)\}^{\nu}\,(\log t)^{\lambda}}{t^{\lambda+\mu+\nu+\beta}}, \; (\lambda > 0,\ \mu > 0,\ \nu > 0,\ \beta > 1).$$

By arbitrary feats of intellectual gymnastics, the reader can easily conceive of any number of other admissible functional forms of U, for which the infinite integral I, *viz.*, (3), represents a continuous and differentiable function of the parameters $(a_1, a_2, \ldots, a_n)$ and admits of indefinite differentiation under the sign of integration *w.r.t.* those parameters.

5. We shall now conclude this paper with a cryptic reference to some of the more common varieties of infinite integrals (involving real positive parameters) which occur frequently in certain branches (Ramsay, 1940; Lamb, 1906) of Applied Mathematics. For the sake of clearness the references are given in a tabular form:—

Infinite Integrals	Remarks
(*i*) $I_1 \equiv (\text{const})\displaystyle\int_d^{\infty}\frac{dt}{\sqrt{\{(a+t)(b+t)(c+t)\}}}$. (Here $n = 3$, $\alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{2}$)	$I_1 =$ the potential of a thin homogeneous homoeoid Γ at a point P, which is interior or exterior to Γ, according as the lower limit d is equal to 0 or the parameter λ of the confocal ellipsoid that goes through P. Evidently I_1 is continuous and differentiable indefinitely *w.r.t.* the parameters a, b, c and the process of differentiation under the sign of integration is valid.

Infinite Integrals	Remarks
(*ii*) $I_2 = \frac{1}{2}\rho(D - Ax^2 - By^2 - Cz^2)$, where $S \equiv \sqrt{\{(a+t)(b+t)(c+t)\}}$, $\frac{1}{2}\rho D = (\text{const}) \int_d^\infty \frac{dt}{S}$, $(n = 3,\ \alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{2})$ $A = (\text{const}) \int_d^\infty \frac{dt}{(a+t)S}$, $(n = 3,\ \alpha_1 = \frac{3}{2},\ \alpha_2 = \alpha_3 = \frac{1}{2})$, with similar values of B, C.	$I_2 =$ the potential of a solid homogeneous ellipsoid Γ at a point $P\ (x, y, z)$, which is interior or exterior to Γ, according as the lower limit is 0 or the parameter λ of the confocal ellipsoid that goes through Γ. As in (*i*), I_2 is continuous and differentiable *w.r.t.* each of the parameters (a, b, c) and differentiation under the integral sign is valid.
(*iii*) $I_3 = (\text{const}) \times x \int_\lambda^\infty \frac{dt}{(a+t)S}$, $(n = 3,\ \alpha_1 = \frac{3}{2},\ \alpha_2 = \alpha_3 = \frac{1}{2},\ d = \lambda)$.	$I_3 =$ the x-component of the attraction of a solid homogeneous ellipsoid at an external point, the parameter of whose confocal is λ; $=$ the velocity-potential of the (irrotational) motion of a liquid, originated by the motion of translation of a solid homogeneous ellipsoid, moving parallel to the x-axis (through the liquid). In either case, I_3 is a continuous and differentiable function of (a, b, c) and differentiation under the sign of integration is permissible.
(*iv*) $I_4 = (\text{const}) yz \int_d^\infty \frac{dt}{(b+t)(c+t)S}$, $(n = 3,\ \alpha_1 = \frac{1}{2},\ \alpha_2 = \alpha_3 = \frac{3}{2};\ d = \lambda)$	$I_4 =$ the velocity potential of the (irrotational) motion of a liquid, produced by the motion of a solid homogeneous ellipsoid, rotating round its x-axis. As in (*i*), (*ii*), (*iii*), differentiation under the sign of integration in (*iv*) is valid.
(*v*) $I_5 = (\text{const}) \int_0^\infty \frac{t\,dt}{(a+t)(b+t)S}$, $(n = 3,\ \alpha_1 = \alpha_2 = \frac{3}{2},\ \alpha_3 = \frac{1}{2},\ f(t) = t)$	$I_5 =$ the square of the angular velocity with which an ellipsoidal mass of homogeneous liquid must rotate round its z-axis so as to maintain *relative* equilibrium, provided that a certain special relation holds amongst the squares of its three semi-axes a, b, c. As in the previous cases, I_5 is a continuous and differentiable function of (a, b, c) and differentiation under the integral sign is legitimate.

N.B.—It is scarcely necessary to add that no two of the parameters a, b, c, appearing in the several infinite integrals, are equal.

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PARALLEL DISPLACEMENT AND SCALAR PRODUCT OF VECTORS—III

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1. This is a continuation of a previous paper (Sen, 1949) in which an attempt was made to study the fundamentals of an arbitrary parallel displacement of vectors defined by

$$dV^l + \Gamma^l_{pq} V^p dx^q = 0 \tag{1.1}$$

in a Riemannian space with the main metric tensor g_{ij} and to establish connections between these displacements and the Levi-Civita parallelism. In the paper referred to we introduced what we called the *conjugate* and the *associate* of (1.1) obtained by replacing Γ^l_{pq} by

$$\Gamma'^l_{pq} = \Gamma^l_{qp} \quad \text{and} \quad \bar{\Gamma}^l_{pq} = \Gamma^l_{pq} + g^{lt} g_{pt,q}$$

respectively, where the notation comma followed by an index denotes the covariant derivative with respect to (1.1). The associate of the associate of (1.1) is (1.1), and for displacements corresponding to Γ^l_{pq} and L^l_{pq}, we have

$$\tfrac{1}{2}(\bar{\Gamma}^l_{pq} + \bar{L}^l_{pq}) = \bar{\Lambda}^l_{pq}, \quad \text{where} \quad \Lambda^l_{pq} = \tfrac{1}{2}(\Gamma^l_{pq} + L^l_{pq}).$$

Obviously, the same thing is true of the conjugate. It was seen that if $\Gamma^l_{pq} = \nabla^l_{pq}$ were the co-efficients of a symmetric affine connection and then if an ordinary bracket followed by an index denoted the covariant derivative with respect to (1.1), the notion of the conjugate and the associate led to the succession of parallelism with the following affine connections:

$$\nabla^l_{pq} = \begin{Bmatrix} l \\ pq \end{Bmatrix} + \tfrac{1}{2} g^{lt}\{(g_{pq})_t - (g_{pt})_q - (g_{qt})_p\} \tag{1.2}$$

$$\Delta^l_{pq} = \begin{Bmatrix} l \\ pq \end{Bmatrix} + \tfrac{1}{2} g^{lt}(g_{pq})_t \tag{1.3}$$

$$\Omega^l_{pq} = \begin{Bmatrix} l \\ pq \end{Bmatrix} - \tfrac{1}{4} g^{lt}\{(g_{pt})_q + (g_{qt})_p \tag{1.4}$$

where, using the notation $\bar{\Gamma}'^l_{pq}$ to represent always the conjugate of the associate (bar followed by dash) of Γ^l_{pq},

$$\nabla^l_{pq} = \tfrac{1}{2}(\bar{\nabla}^l_{pq} + \bar{\nabla}'^l_{pq}), \quad \Omega^l_{pq} = \tfrac{1}{2}(\bar{\Delta}^l_{pq} + \bar{\Delta}'^l_{pq}) \quad \text{and} \quad \begin{Bmatrix} l \\ pq \end{Bmatrix}$$

is the Christoffel symbol. Whence followed

$$2\begin{Bmatrix} l \\ pq \end{Bmatrix} + \nabla^l_{pq} - \Delta^l_{pq} - 2\Omega^l_{pq} = 0. \tag{1.5}$$

It was also seen that the covariant derivatives of g_{pq} with respect to the parallel displacements corresponding to (1.3) and (1.4) were respectively

$$-\tfrac{1}{2}\{(g_{pt})_q+(g_{qt})_p\} \quad \text{and} \quad \tfrac{1}{2}(g_{pq})_t-\tfrac{1}{4}\{(g_{pt})_q+(g_{qt})_p\} \tag{1·6}$$

and that some geometrical interpretations involving changes in the scalar product of vectors were derived from these derivatives.

Now (1.5) expresses the Christoffel symbol in terms of other symmetric affine connections. In this paper we propose to study the nature and relations of curvature tensors formed with these and other affine connections that would necessarily arise. It is shown that the curvature tensor formed with the Christoffel symbol can be expressed linearly, in various ways, in terms of other curvature tensors.

2. We require the following two formulæ. If Γ^l_{pq} and L^l_{pq} are arbitrary affine connections and

$$\Gamma^l_{pq}-L^l_{pq} = T^l_{pq}, \quad \tfrac{1}{2}(\Gamma^l_{pq}+L^l_{pq}) = \Lambda^l_{pq}$$

and if Γ^t_{ijk}, L^t_{ijk}, Λ^t_{ijk} are the corresponding curvature tensors, then

$$L^t_{ijk}-\Gamma^t_{ijk} = T^t_{ij,k}-T^t_{ik,j}+T^t_{sj}T^s_{ik}-T^t_{sk}T^s_{ij}-T^t_{is}(\Gamma^s_{jk}-\Gamma^s_{kj}) \tag{2·1}$$

$$\Lambda^t_{ijk}+\tfrac{1}{2}(\Gamma^t_{ijk}+L^t_{ijk}) = \tfrac{1}{4}(T^t_{sk}T^s_{ij}-T^t_{sj}T^s_{ik}) \tag{2.2}$$

Now let K^t_{ijk}, ∇^t_{ijk}, Δ^t_{ijk} and Ω^t_{ijk} be the curvature tensors formed with the Christoffel symbol and (1.2), (1.3), (1,4) respectively. Without going into the laborious details of working out, we merely state the following relations which are obtained from (1.2), (1.3), (1.4) by the application of (2.1) and the use of (1.6)

$$\begin{aligned} K^t_{ijk}-\nabla^t_{ijk} = {} & \tfrac{1}{2}g^{st}[\{(g_{ij})_{sk}-(g_{is})_{jk}-(g_{js})_{ik}\}-\{(g_{ik})_{sj}-(g_{is})_{kj}-(g_{ks})_{ij}\}] \\ & +\tfrac{1}{4}g^{st}g^{mn}[\{(g_{ij})_n-(g_{in})_j-(g_{jn})_i\}\{(g_{ks})_m-(g_{km})_s-(g_{sm})_k\} \\ & -\{(g_{ik})_n-(g_{in})_k-(g_{kn})_i\}\{(g_{js})_m-(g_{jm})_s-(g_{sm})_j\}] \end{aligned} \tag{2·3}$$

$$\begin{aligned} K^t_{ijk}-\Delta^t_{ijk} = {} & \tfrac{1}{2}g^{st}[(g_{ij})_{sk}-(g_{ik})_{sj}]+\tfrac{1}{4}g^{st}g^{mn}[(g_{ij})_n\{(g_{ks})_m-(g_{km})_s-(g_{sm})_k\} \\ & +\{(g_{in})_j+(g_{jn})_i\}(g_{km})_s-(g_{ik})_n\{(g_{js})_m-(g_{jm})_s-(g_{sm})_j\}-\{(g_{in})_k+(g_{kn})_i\}(g_{jm})_s] \end{aligned} \tag{2.4}$$

$$\begin{aligned} K^t_{ijk}-\Omega^t_{ijk} = {} & -\tfrac{1}{4}g^{st}[(g_{is})_{jk}+(g_{js})_{ik}-(g_{is})_{kj}-(g_{ks})_{ij}] \\ & +\tfrac{1}{8}g^{st}g^{mn}[(g_{ij})_n\{(g_{ks})_m+(g_{sm})_k\}-\{(g_{in})_j+(g_{jn})_i\}\{(g_{ks})_m-(g_{km})_s-(g_{sm})_k\} \\ & -(g_{ik})_n\{(g_{js})_m+(g_{sm})_j\}+\{(g_{in})_k+(g_{kn})_i\}\{(g_{js})_m-(g_{jm})_s-(g_{sm})_j\}] \\ & -\tfrac{1}{16}g^{st}g^{mn}[\{(g_{in})_j+(g_{jn})_i\}\{(g_{ks})_m+(g_{sm})_k\}-\{(g_{in})_k+(g_{kn})_i\}\{(g_{js})_m+(g_{sm})_j\}]. \end{aligned} \tag{2.5}$$

Therefore, from (2.3), (2.4), (2.5) we obtain the following relation corresponding to (1 5):

$$\begin{aligned} 2K^t_{ijk}+\nabla^t_{ijk}-\Delta^t_{ijk}-2\Omega^t_{ijk} = {} & \tfrac{1}{4}g^{st}g^{mn}[(g_{ij})_n\{(g_{ks})_m+(g_{sm})_k\}+\{(g_{in})_j+(g_{jn})_i\}(g_{km})_s \\ & -(g_{ik})_n\{(g_{js})_m+(g_{sm})_j\}-\{(g_{in})_k+(g_{kn})_i\}(g_{jm})_s] \\ & -\tfrac{1}{8}g^{st}g^{mn}[\{(g_{in})_j+(g_{jn})_i\}\{(g_{ks})_m+(g_{sm})_k\}-\{(g_{in})_k+(g_{kn})_i\}\{(g_{js})_m+(g_{sm})_j\}] \end{aligned} \tag{2.6}$$

Also, besides the affine connections (1.2), (1.3), (1.4), we require the following ones:

$$\omega^l_{pq} = \tfrac{1}{2}(\nabla^l_{pq} + \Delta^l_{pq}) = \begin{Bmatrix} l \\ pq \end{Bmatrix} + \tfrac{1}{2}g^{lt}(g_{pq})_t - \tfrac{1}{4}g^{lt}\{(g_{pt})_q + (g_{qt})_p\} \tag{2.7}$$

$$\theta^l_{pq} = \tfrac{1}{2}(\Delta^l_{pq} + \Omega^l_{pq}) = \begin{Bmatrix} l \\ pq \end{Bmatrix} + \tfrac{1}{4}g^{lt}(g_{pq})_t - \tfrac{1}{8}g^{lt}\{(g_{pt})_q + (g_{qt})_p\} \tag{2.8}$$

$$\varphi^l_{pq} = \tfrac{1}{2}(\nabla^l_{pq} + \omega^l_{pq}) = \begin{Bmatrix} l \\ pq \end{Bmatrix} + \tfrac{1}{2}g^{lt}(g_{pq})_t - \tfrac{3}{8}g^{lt}\{g_{pt})_q + (g_{qt})_p\} \tag{2.9}$$

$$\psi^l_{pq} = \tfrac{1}{2}(\Omega^l_{pq} + \omega^l_{pq}) = \begin{Bmatrix} l \\ pq \end{Bmatrix} + \tfrac{1}{4}g^{lt}\{(g_{pq})_t - (g_{pt})_q - (g_{qt})_p\} \tag{2.10}$$

$$\chi^l_{pq} = \tfrac{1}{2}(\theta^l_{pq} + \varphi^l_{pq}) = \begin{Bmatrix} l \\ pq \end{Bmatrix} + \tfrac{3}{8}g^{lt}(g_{pq})_t - \tfrac{1}{8}g^{lt}\{(g_{pt})_q + (g_{qt})_p\} \tag{2.11}$$

Let ω^t_{ijk}, θ^t_{ijk} φ^t_{ijk}, ψ^t_{ijk}, χ^t_{ijk} be the curvature tensors formed with (2.7) to (2.11). Then, since

$$\varphi^l_{pq} - \theta^l_{pq} = \tfrac{1}{4}g^{lt}\{(g_{pq})_t - (g_{pt})_q - (g_{qt})_p\},$$

$$\Delta^l_{pq} - \nabla^l_{pq} = \tfrac{1}{2}g^{lt}\{(g_{pt})_q + (g_{qt})_p\}, \quad \omega^l_{pq} - \Omega^l_{pq} = \tfrac{1}{2}g^{lt}(g_{pq})_t,$$

we have by (2.2)

$$\begin{aligned}\omega^t_{ijk} - \tfrac{1}{2}(\nabla^t_{ijk} + \Delta^t_{ijk}) = \tfrac{1}{16}g^{st}g^{mn}[\{(g_{in})_j + (g_{jn})_i\}\{(g_{ks})_m + (g_{sm})_k\} \\ - \{(g_{in})_k + (g_{kn})_i\}\{(g_{js})_m + (g_{sm})_j\}]\end{aligned} \tag{2.12}$$

$$\psi^t_{ijk} - \tfrac{1}{2}(\Omega^t_{ijk} + \omega^t_{ijk}) = \tfrac{1}{16}g^{st}g^{mn}[(g_{ij})_n(g_{km})_s - (g_{ik})_n(g_{jm})_s] \tag{2.13}$$

$$\begin{aligned}\chi^t_{ijk} - \tfrac{1}{2}(\theta^t_{ijk} + \varphi^t_{ijk}) = \tfrac{1}{64}g^{st}g^{mn}[\{(g_{ij})_n - (g_{in})_j - (g_{jn})_i\}\{(g_{km})_s - (g_{ks})_m - (g_{sm})_k\} \\ - \{(g_{ik})_n - (g_{in})_k - (g_{kn})_i\}\{(g_{jm})_s - (g_{js})_m - (g_{sm})_j\}].\end{aligned} \tag{2.14}$$

The expression within the square bracket on the right-hand side of (2.14)

$$\begin{aligned}= -[(g_{ij})_n\{(g_{ks})_m + (g_{sm})_k\} + \{(g_{in})_j + (g_{jn})_i\}(g_{km})_s - (g_{ij})_n(g_{km})_s \\ - (g_{ik})_n\{(g_{js})_m + (g_{sm})_j\} - \{(g_{in})_k + (g_{kn})_i\}(g_{jm})_s + (g_{ik})_n(g_{jm})_s] \\ + [\{(g_{in})_j + (g_{jn})_i\}\{(g_{ks})_m + (g_{sm})_k\} - \{(g_{in})_k + (g_{kn})_i\}\{(g_{js})_m + (g_{sm})_j\}]\end{aligned}$$

Therefore, using (2.12), (2.13), (2.14) in (2.6) we get

$$2K^t_{ijk} + \nabla^t_{ijk} - \Delta^t_{ijk} - 2\Omega^t_{ijk} = 2\{\omega^t_{ijk} - \tfrac{1}{2}(\nabla^t_{ijk} + \Delta^t_{ijk})\} + 4\{\psi^t_{ijk} - \tfrac{1}{2}(\Omega^t_{ijk} + \omega^t_{ijk})\} - 16\{\chi^t_{ijk} - \tfrac{1}{2}(\theta^t_{ijk} + \varphi^t_{ijk})\}.$$

Hence finally we have the following result

$$K^t_{ijk} + \nabla^t_{ijk} - 4\theta^t_{ijk} - 4\varphi^t_{ijk} - 2\psi^t_{ijk} + 8\chi^t_{ijk} = 0. \tag{2.15}$$

In this formula K^t_{ijk} has been expressed in terms of ∇^t_{ijk} and four other curvature tensors. All these five curvature tensors are ultimately derived from ∇^l_{pq} with which we have started.

If, however, we start with both ∇^l_{pq} and $\left\{ {l \atop pq} \right\}$ instead of ∇^l_{pq} only and put

$$\xi^l_{pq} = \tfrac12\left(\left\{ {l \atop pq} \right\} + \nabla^l_{pq}\right), \quad \eta^l_{pq} = \tfrac12\left(\left\{ {l \atop pq} \right\} + \Delta^l_{pq}\right), \quad \zeta^l_{pq} = \tfrac12\left(\left\{ {l \atop pq} \right\} + \Omega^l_{pq}\right)$$

then since

$$\left.\begin{aligned} \xi^t_{ijk} - \tfrac12(K^t_{ijk} + \nabla^t_{ijk}) &= 4\{\chi^t_{ijk} - \tfrac12(\theta^t_{ijk} + \varphi^t_{ijk})\} \\ \eta^t_{ijk} - \tfrac12(K^t_{ijk} + \Delta^t_{ijk}) = \psi^t_{ijk} - \tfrac12(\Omega^t_{ijk} + \omega^t_{ijk}), \quad 4\{\zeta^t_{ijk} - \tfrac12(K^t_{ijk} + \Omega^t_{ijk})\} &= \omega^t_{ijk} - \tfrac12(\nabla^t_{ijk} + \Delta^t_{ijk}) \end{aligned}\right\} \quad (2.16)$$

we obtain the formula

$$6K^t_{ijk} - \nabla^t_{ijk} + \Delta^t_{ijk} + 2\Omega^t_{ijk} + 4\xi^t_{ijk} - 4\eta^t_{ijk} - 8\zeta^t_{ijk} = 0. \quad (2.17)$$

3. We may also express K^t_{ijk} involving curvature tensors formed with respect to the associates and the conjugates. For this purpose we have to modify the procedure of the last article as follows:

By (2.1) we have

$$\bar\nabla^t_{ijk} - \nabla^t_{ijk} = -g^{st}[(g_{is})_{jk} - (g_{is})_{kj}]$$

$$\bar\nabla'^t_{ijk} - \Delta^t_{ijk} = -g^{st}[(g_{js})_{ik} - (g_{ks})_{ij}] + g^{st}g^{mn}[(g_{jn})_i\{(g_{ks})_m - (g_{sm})_k\} - (g_{kn})_i\{(g_{js})_m - (g_{sm})_j\}]$$

$$\begin{aligned} \therefore \quad \nabla^t_{ijk} - \tfrac12(\bar\nabla^t_{ijk} + \bar\nabla'^t_{ijk}) &= \tfrac12 g^{st}[(g_{is})_{jk} + (g_{js})_{ik} - (g_{is})_{kj} - (g_{ks})_{ij}] \\ &\quad - \tfrac12 g^{st}g^{mn}[(g_{jn})_i\{(g_{ks})_m - (g_{sm})_k\} - (g_{kn})_i\{(g_{js})_m - (g_{sm})_j\}] \quad (3.1) \end{aligned}$$

In the same way

$$\begin{aligned} \Delta^t_{ijk} - \tfrac12(\bar\Delta^t_{ijk} + \bar\Delta'^t_{ijk}) = {} & -\tfrac12 g^{st}[\{2(g_{ij})_{sk} + (g_{is})_{jk} + (g_{js})_{ik}\} - \{2(g_{ik})_{sj} + (g_{is})_{kj} + (g_{ks})_{ij}\}] \\ & - \tfrac14 g^{st}g^{mn}[(g_{ij})_n\{(g_{ks})_m - 2(g_{km})_s - 3(g_{sm})_k\} - (g_{ik})_n\{(g_{js})_m - 2(g_{jm})_s - 3(g_{sm})_j\} \\ & + (g_{in})_j\{2(g_{ks})_m + (g_{km})_s - (g_{sm})_k\} - (g_{in})_k\{2(g_{js})_m + (g_{jm})_s - (g_{sm})_j\} \\ & + (g_{jn})_i\{(g_{ks})_m + (g_{km})_s\} - (g_{kn})_i\{(g_{js})_m + (g_{jm})_s\}] \quad (3.2) \end{aligned}$$

Also, by (2.2) we have

$$\begin{aligned} \Delta^t_{ijk} - \tfrac12(\bar\nabla^t_{ijk} + \bar\nabla'^t_{ijk}) &= 4\{\Omega^t_{ijk} - \tfrac12(\bar\Delta^t_{ijk} + \bar\Delta'^t_{ijk})\} \\ &= \tfrac14 g^{st}g^{mn}[\{(g_{in})_j - (g_{jn})_i\}\{(g_{sm})_k - (g_{ks})_m\} - \{(g_{in})_k - (g_{kn})_i\}\{(g_{sm})_j - (g_{js})_m\}] \quad (3.3) \end{aligned}$$

Therefore, from (2.4), (3.1), (3.2) we have

$$\begin{aligned} 2K^t_{ijk} &+ \nabla^t_{ijk} - \tfrac12(\bar\nabla^t_{ijk} + \bar\nabla'^t_{ijk}) - (\bar\Delta^t_{ijk} + \bar\Delta'^t_{ijk}) \\ &= 2(K^t_{ijk} - \Delta^t_{ijk}) + \{\nabla^t_{ijk} - \tfrac12(\bar\nabla^t_{ijk} + \bar\nabla'^t_{ijk})\} + 2\{\Delta^t_{ijk} - \tfrac12(\bar\Delta^t_{ijk} + \bar\Delta'^t_{ijk})\} \\ &= \tfrac14 g^{st}g^{mn}[(g_{ij})_n\{(g_{ks})_m + (g_{sm})_k\} - (g_{ik})_n\{(g_{js})_m + (g_{sm})_j\} \\ &\quad + (g_{in})_j\{(g_{km})_s + (g_{sm})_k - 2(g_{ks})_m\} - (g_{in})_k\{(g_{jm})_s + (g_{sm})_j - 2(g_{js})_m\} \\ &\quad + (g_{jn})_i\{(g_{km})_s + (g_{ks})_m - 2(g_{sm})_k\} - (g_{kn})_i\{(g_{jm})_s + (g_{js})_m - 2(g_{sm})_j\}] \quad (3.4) \end{aligned}$$

Again, besides the affine connections

$$\bar{\nabla}^l_{pq} = \begin{Bmatrix} l \\ pq \end{Bmatrix} + \tfrac{1}{2} g^{lt}\{(g_{pq})_t + (g_{pt})_q - (g_{qt})_p\} \tag{3.5}$$

$$\bar{\nabla}'^l_{pq} = \begin{Bmatrix} l \\ pq \end{Bmatrix} + \tfrac{1}{2} g^{lt}\{(g_{pq})_t - (g_{pt})_q + (g_{pt})_p\} \tag{3.6}$$

$$\bar{\Delta}^l_{pq} = \begin{Bmatrix} l \\ pq \end{Bmatrix} - \tfrac{1}{2} g^{lt}(g_{qt})_p \tag{3.7}$$

$$\bar{\Delta}'^l_{pq} = \begin{Bmatrix} l \\ pq \end{Bmatrix} - \tfrac{1}{2} g^{lt}(g_{pt})_q \tag{3.8}$$

we require the following ones:

$$\alpha^l_{pq} = \tfrac{1}{2}(\nabla^l_{pq} + \bar{\nabla}^l_{pq}) = \begin{Bmatrix} l \\ pq \end{Bmatrix} + \tfrac{1}{2} g^{lt}\{(g_{pq})_t - (g_{qt})_p\} \tag{3.9}$$

$$\beta^l_{pq} = \tfrac{1}{2}(\nabla^l_{pq} + \bar{\nabla}'^l_{pq}) = \begin{Bmatrix} l \\ pq \end{Bmatrix} + \tfrac{1}{2} g^{lt}\{(g_{pq})_t - (g_{pt})_q\} \tag{3.10}$$

$$\gamma^l_{pq} = \tfrac{1}{2}(\bar{\nabla}^l_{pq} + \bar{\Delta}^l_{pq}) = \begin{Bmatrix} l \\ pq \end{Bmatrix} + \tfrac{1}{4} g^{lt}\{(g_{pq})_t + (g_{pt})_q\} - \tfrac{1}{2} g^{lt}(g_{qt})_p \tag{3.11}$$

$$\delta^l_{pq} = \tfrac{1}{2}(\bar{\nabla}'^l_{pq} + \bar{\Delta}'^l_{pq}) = \begin{Bmatrix} l \\ pq \end{Bmatrix} + \tfrac{1}{4} g^{lt}\{(g_{pq})_t + (g_{qt})_p\} - \tfrac{1}{2} g^{lt}(g_{pt})_q, \tag{3.12}$$

and also $\omega^l_{pq} = \tfrac{1}{2}(\alpha^l_{pq} + \beta^l_{pq})$ and ψ^l_{pq} as given by (2.7) and (2.10). Since

$$\bar{\nabla}^l_{pq} - \nabla^l_{pq} = \tfrac{1}{2} g^{lt}(g_{pt})_q, \quad \bar{\nabla}^l_{pq} - \bar{\Delta}^l_{pq} = \tfrac{1}{2} g^{lt}\{(g_{pq})_t + (g_{pt})_q\},$$

$$\bar{\nabla}'^l_{pq} - \nabla^l_{pq} = \tfrac{1}{2} g^{lt}(g_{qt})_p, \quad \bar{\nabla}'^l_{pq} - \bar{\Delta}'^l_{pq} = \tfrac{1}{2} g^{lt}\{(g_{pq})_t + (g_{qt})_p,$$

by the application (2.2) it follows from (3.9) to (3.12) that

$$\alpha^t_{ijk} - \tfrac{1}{2}(\nabla^t_{ijk} + \bar{\nabla}^t_{ijk}) = \tfrac{1}{16} g^{st} g^{mn}[(g_{in})_j (g_{sm})_k - (g_{in})_k (g_{sm})_j] \tag{3.13}$$

$$\beta^t_{ijk} - \tfrac{1}{2}(\nabla^t_{ijk} + \bar{\nabla}'^t_{ijk}) = \tfrac{1}{16} g^{st} g^{mn}[(g_{jn})_i (g_{ks})_m - (g_{kn})_i (g_{js})_m] \tag{3.14}$$

$$\gamma^t_{ijk} - \tfrac{1}{2}(\bar{\nabla}^t_{ijk} + \bar{\Delta}^t_{ijk}) = \tfrac{1}{16} g^{st} g^{mn}[\{(g_{ij})_n + (g_{in})_j\}\{(g_{km})_s + (g_{sm})_k\} - \{(g_{ik})_n + (g_{in})_k\}\{(g_{jm})_s + (g_{sm})_j\}] \tag{3.15}$$

$$\delta^t_{ijk} - \tfrac{1}{2}(\bar{\nabla}'^t_{ijk} + \bar{\Delta}'^t_{ijk}) = \tfrac{1}{16} g^{st} g^{mn}[\{(g_{ij})_n + (g_{jn})_i\}\{(g_{km})_s + (g_{ks})_m\} - \{(g_{ik})_n + (g_{kn})_i\}\{(g_{jm})_s + (g_{js})_m\}]. \tag{3.16}$$

Therefore using (3.13) to (3.16) and (2.13), (3.3) in (3.4) we get

$$\begin{aligned} &2K^t_{ijk} + \nabla^t_{ijk} - \tfrac{1}{2}(\bar{\nabla}^t_{ijk} + \bar{\nabla}'^t_{ijk}) - (\bar{\Delta}^t_{ijk} + \bar{\Delta}'^t_{ijk}) \\ &\quad = 4\{\gamma^t_{ijk} - \tfrac{1}{2}(\bar{\nabla}^t_{ijk} + \bar{\Delta}^t_{ijk})\} + 4\{\delta^t_{ijk} - \tfrac{1}{2}(\bar{\nabla}'^t_{ijk} + \bar{\Delta}'^t_{ijk})\} - 8\{\alpha^t_{ijk} - \tfrac{1}{2}(\nabla^t_{ijk} + \bar{\nabla}^t_{ijk})\} \\ &\qquad - 8\{\beta^t_{ijk} - \tfrac{1}{2}(\nabla^t_{ijk} + \bar{\nabla}'^t_{ijk})\} + 8\{\Omega^t_{ijk} - \tfrac{1}{2}(\bar{\Delta}^t_{ijk} + \bar{\Delta}'^t_{ijk})\} - 8\{\psi^t_{ijk} - \tfrac{1}{2}(\omega^t_{ijk} + \Omega^t_{ijk})\}. \end{aligned}$$

Hence finallly we obtain the following result:

$$4K^t_{ijk}-6\nabla^t_{ijk}-3(\bar{\nabla}^t_{ijk}+\dot{\nabla}'^t_{ijk})+8(\sigma^t_{ijk}+\beta^t_{ijk})-4\omega^t_{ijk}$$

$$+4(\bar{\Delta}^t_{ijk}+\bar{\Delta}'^t_{ijk})-12\Omega^t_{ijk}-4(\gamma^t_{ijk}+\delta^t_{ijk})+8\psi^t_{ij}=0 \quad (3.17)$$

As in the case of (2.15), the formula (3.17) expresses K^t_{ijk} in terms of curvature tensors all of which are ultimately derivable from the arbitrary ∇^l_{pq} with which we have started. And a result corresponding to (2.17) can in this case also be obtained if we start with both ∇^l_{pq} and $\left\{ {l \atop pq} \right\}$ instead of ∇^l_{pq} only.

4. The tensor $g_{st}K^t_{ijk}=K_{sijk}$ is known to possess the following properties:

$$K_{sijk}+K_{sikj}=0, \quad K_{sijk}+K_{isjk}=0, \quad K_{sijk}=K_{jksi}, \quad K_{sijk}+K_{sjki}+K_{skij}=0 \quad (4.1)$$

From the nature of the tensors that we have obtained so far, we note that the following are some of the tensors which possess all the properties (4.1)

(i) $g^{mn}[(g_{ij})_n(g_{ks})_m-(g_{ik})_n(g_{js})_m]$

(ii) $g^{mn}[\{(g_{in})_j+(g_{jn})_i\}\{(g_{km})_s+(g_{sm})_k\}-\{(g_{in})_k+(g_{kn})_i\}\{(g_{jm})_s+(g_{sm})_j\}]$

(iii) $g^{mn}[(g_{ij})_n\{(g_{km})_s+(g_{sm})_k\}+\{(g_{in})_j+(g_{jn})_i\}(g_{ks})_m$
$-(g_{ik})_n\{(g_{jm})_s+(g_{sm})_j\}-\{(g_{in})_k+(g_{kn})_i\}(g_{js})_m]$

(iv) $g^{mn}[(g_{ij})_n\{(g_{ks})_m-(g_{km})_s-(g_{sm})_k\}+\{(g_{ij})_n-(g_{in})_j-(g_{jn})_i\}(g_{ks})_m$
$-(g_{ik})_n\{(g_{js})_m-(g_{jm})_s-(g_{sm})_j\}-\{(g_{ik})_n-(g_{in})_k-(g_{kn})_i\}(g_{js})_m]$.

Obviously, linear combinations of these tensors are tensors of the same kind. One such tensor is

$$P_{sijk}=\tfrac{1}{4}g^{mn}[\{(g_{ij})_n-(g_{in})_j-(g_{jn})_i\}\{(g_{ks})_m-(g_{km})_s-(g_{sm})_k\}$$
$$-\{(g_{ik})_n-(g_{in})_k-(g_{kn})_i\}\{(g_{js})_m-(g_{jm})_s-(g_{sm})_j\}]$$

which occurs in the second part of the right-hand side of (2.8). This tensor may be split up into two parts

$$B_{sijk}=\tfrac{1}{8}g^{mn}[\{(g_{ij})_n-(g_{in})_j-(g_{jn})_i\}\{(g_{ks})_m-2(g_{km})_s\}-\{(g_{ik})_n-(g_{in})_k-(g_{kn})_i\}\{(g_{js})_m-2(g_{jm})_s\}]$$

$$C_{sijk}=\tfrac{1}{8}g^{mn}[\{(g_{ij})_n-(g_{in})_j-(g_{jn})_i\}\{(g_{ks})_m-2(g_{sm})_k\}-\{(g_{ik})_n-(g_{in})_k-(g_{kn})_i\}\{(g_{js})_m-2(g_{sm})_j\}]$$

such that

$$B_{sijk}+B_{iskj}+B_{jksi}+B_{kjis}=C_{sijk}+C_{iskj}+C_{jksi}+C_{kjis}=2P_{sijk}. \quad (4.2)$$

In this connection we may notice the following relation. If the crooked bracket followed by an index denotes the covariant derivative with respect to the Levi-Civita parallelism, then

$$\{(g_{ij})_s\}_k = (g_{ij})_{sk} + \tfrac{1}{2}g^{mn}[(g_{jm})_s\{(g_{ik})_n - (g_{kn})_i - (g_{in})_k\} + (g_{im})_s\{(g_{jk})_n - (g_{kn})_j - (g_{jn})_k\} + (g_{ij})_n\{(g_{sk})_m - (g_{km})_s - (g_{sm})_k\}].$$

Therefore Σ_{sijk}, defined as follows, has the following value:

$$\begin{aligned}\Sigma_{sijk} &= [\{(g_{ij})_s\}_k - \{(g_{is})_j\}_k - \{(g_{js})_i\}_k] - [\{(g_{ik})_s\}_j - \{(g_{is})_k\}_j - \{(g_{ks})_i\}_j] \\ &= [\{(g_{ij})_{sk} - (g_{is})_{jk} - (g_{js})_{ik}\} - \{(g_{ik})_{sj} - (g_{is})_{kj} - (g_{ks})_{ij}\}] + g^{mn}[\{(g_{ij})_n - (g_{in})_j - (g_{jn})_i\} \\ &\qquad \times \{(g_{ks})_m - (g_{km})_s\} - \{(g_{ik})_n - (g_{in})_k - (g_{kn})_i\}\{(g_{js})_m - (g_{jm})_s\}] \qquad (4.3)\end{aligned}$$

Hence, from (2 3), (2.14), (2.16) and (4.3) we have

$$2(K_{sijk} - \nabla_{sijk}) = \Sigma_{sijk} + 8\{\xi_{sijk} - \tfrac{1}{2}(K_{sijk} + \nabla_{sijk})\}$$

or,

$$\Sigma_{sijk} = 6K_{sijk} + 2\nabla_{sijk} - 8\xi_{sijk} \qquad (4.4)$$

Finally, we observe that (2.3) expresses ∇^t_{ijk} in combination with displacement with respect to Levi-Civita parallelism. We may however express ∇^t_{ijk} directly in terms of the parallelism corresponding to (1.2) as follows: We introduce orthogonal ennuple defined, as usual, by

$$g_{ij} = \sum_T {}^T h_i {}^T h_j, \quad {}_T h^j = g^{ij}\,{}^T h_i.$$

We have

$${}_T h^l\{{}^T h_p\}_q = \Gamma^l_{pq} - \begin{Bmatrix} l \\ pq \end{Bmatrix}, \text{ where } \Gamma^l_{pq} = {}_T h^l \frac{\partial\,{}^T h_p}{\partial x^q}$$

is the affine connection corresponding to Einstein's teleparallelism. Also, by (1.2)

$${}_T h^l\{{}^T h_p\}_q = {}_T h^l({}^T h_p)_q + \tfrac{1}{2}g^{lt}\{(g_{pq})_t - (g_{pt})_q - (g_{qt})_p\}.$$

Therefore

$$\nabla^l_{pq} - \Gamma^l_{pq} = -{}_T h^l({}^T h_p)_q. \qquad (4.5)$$

As the curvature tensor Γ^t_{ijk} now vanishes, we get from (4.5) by the application of (2.1)

$$\nabla^t_{ijk} = {}_T h^t\{({}^T h_i)_{kj} - ({}^T h_i)_{jk}\} + ({}^T h_i)_k({}_T h^t)_j - ({}^T h_i)_j({}_T h^t)_k + g^{st}\{({}^T h_s)_j({}^T h_i)_k - ({}^T h_s)_k({}^T h_i)_j\}. \qquad (4.6)$$

In conclusion, it may be mentioned that one may continue the process by which Δ^l_{pq} is constructed from ∇^l_{pq} and Ω^l_{pq} from Δ^l_{pq}, and obtain new symmetric connections, although it has not been necessary for our purpose to do so. From the implications of the results obtained so far, it is believed that the study of Riemannian geometry with the help of an arbitrary parallel displacement together with the notions of the associate and the conjugate would reveal many interesting properties. Various conditions may also be imposed on the parallelism: for example, the condition satisfied by the Levi-Civita parallelism (Levi-Civita, 1927), namely the parallelism with symmetric connection for which the scalar product remains invariant, or the condition imposed on the scalar product by Weyl (Weyl, 1921)) or a certain condition considered by the author (Sen, 1948).

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A NOTE ON CONDITIONAL TESTS OF SIGNIFICANCE

By

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1. Introduction

Let $x_1, x_2, \ldots, x_p, y_1, y_2, \ldots, y_n$ be $(p+n)$ random variables whose joint distribution involving $(k+l)$ parameters is given by :

$$\varphi(x_1, \ldots, x_p, y_1, \ldots, y_n/\theta_1, \ldots, \theta_k, \theta_{k+1}, \ldots, \theta_{k+l})\prod dx\prod dy \tag{1}$$

Denoting by x, y, θ, θ' the respective vectors $(x_1, \ldots, x_p)$, $(y_1, \ldots, y_n)$, $(\theta_1, \ldots, \theta_k)$, $(\theta_{k+1}, \ldots, \theta_{k+l})$ the probability density $(p.\ d.)$ in (1) can be written as:

$$\varphi = \varphi_1(x, \theta)\varphi_2(y/x, \theta') \tag{2}$$

where $\varphi_2(y/x, \theta')$ stands for the conditional distribution of y for given x; $\varphi_1(x, \theta)$ is the unconditional distribution of x; and we assume further that the parameters θ, θ' separate out in the two distributions φ_1 and φ_2 as indicated in (2). Under the last condition it is possible to test any hypothesis regarding θ' starting with either the $p.\ d.\ \varphi$ or the conditional $p.\ d.\ \varphi_2$. All the tests of significance connected with multiple regression or correlation or even contingency χ^2 are based on the conditional $p.\ d.$ Obviously the optimum test criteria obtained from φ or φ_2 are not equivalent in general, and it is the purpose of the note to study the conditions under which an optimum test based on φ_2 will retain its optimum properties also with regard to φ.

2. Tests of Significance Independent of φ_1

Let us consider the most powerful critical region w_φ and w_{φ_2} for testing the null hypothesis $H_0(\theta')$ against the alternative $H_1(\theta')$ based on respectively the unconditional $p.\ d.\ \varphi$ and the conditional $p.\ d.\ \varphi_2$. In that case the following relation will hold inside w_φ or w_{φ_2}

$$\varphi_2(y/x, H_1) \geq k\varphi_2(y/x, H_0) \tag{3}$$

the constant k being however chosen in different ways for the two cases.

For w_φ, k_φ satisfies the equation

$$\int_{w_\varphi} \varphi(H_0)\prod dx\prod dy = \alpha \tag{4}$$

whereas for w_{φ_2}, k_{φ_2} is determined by the relation.

$$\int_{w_{\varphi_2}} \varphi_2(H_0)\prod dy = \alpha. \tag{5}$$

Though we consider for convenience a simple hypothesis against a simple alternative, it is not difficult to see that for composite hypotheses concerning θ' or for such hypotheses against sets of alternatives a relation similar to (3) *involving only* φ_2 will be obtained in the cases of both unconditional or conditional probability.

Let us suppose that the ratio $\varphi_2(y/x, H_1)/\varphi_2(y/x, H_0)$ gives rise to a statistic T such that

$$\text{Inside } w_\varphi, \quad T \geq \lambda_\varphi \tag{6}$$

$$\text{Inside } w_{\varphi_2}, \quad T \geq \lambda_{\varphi_2}. \tag{7}$$

The difference between the two tests (6) and (7) arises out of the fact that λ_φ does not involve the random variables x while λ_{φ_2} possibly depends on x. That these two tests will be equivalent independently of φ_1 is contained in the following theorem:

Theorem. *The necessary and sufficient condition that* $w_\varphi = w_{\varphi_2}$ *independently of* φ_1, *i.e., the distribution of* x, *is that the statistic* T *is distributed independently of* $x_1, x_2, \ldots, x_p$ *when the null hypothesis is true.*

To prove the sufficiency of the condition, we have from (4) and (6)

$$\int_{w_\varphi} \varphi(H_0)\prod dx \prod dy = \int_{T>\lambda_\varphi} \varphi(H_0)\prod dx \prod dy = \alpha. \tag{8}$$

Since T and x are independently distributed, (8) will be equal to.

$$\int_{T>\lambda_\varphi} \varphi_1(x)f(T)dT\prod dx = \int_{T>\lambda_\varphi} f(T)dT = \alpha. \tag{9}$$

Again from (5) and (7)

$$\int_{w_{\varphi_2}} \varphi_2(H_0)\prod dy = \int_{T>\lambda_{\varphi_2}} \varphi_2(H_0)\prod dy = \int_{T>\lambda_{\varphi_2}} f(T)dT = \alpha. \tag{10}$$

From (9) and (10) it follows that $\lambda_\varphi = \lambda_{\varphi_2}$ which proves that $w_\varphi \equiv w_{\varphi_2}$.

To prove the necessary part of the theorem it is enough to show that in case T is not distributed independently of x there exists a φ_1 such that the conditional distribution of T is different from the unconditional distribution of T. This is, of course, obvious, for if φ_1 is taken to be unity, *i.e.*, $x_1, x_2, \ldots, x_p$ are distributed independently and uniformly, the conditional distribution of T is, say,

$$f(T/x, \theta')dT \tag{11}$$

while the unconditional distribution of T will be

$$dT\int f(T/x, \theta')\prod dx = \psi(T)dT \text{ say} \tag{12}$$

Hence the theorem is proved.

3. Power Function of the Conditional Tests

The importance of the theorem proved in Section 2 lies in the fact that when optimum critical regions are of the type described above, the knowledge regarding the distribution of x, *i.e.*, φ_1 cannot help to improve the choice of a critical region. But for planning an experiment we need not only a powerful test but also to know the exact power function of the test. The power of the test will depend in general on the distribution of the conditioning variables x even when an optimum test independent of φ_1 exists.

If, however, the optimum test is such that the distribution of the corresponding statistic is independent of φ_1 both on the null hypothesis as well as on the alternative hypothesis, the test can be truly said to be ***completely independent of the distributian of the conditioning variables.*** Such a situation obtains in the case of tests for partial correlation coefficients. Though the tests for partial regression, multiple correlation, etc., are not completely independent of φ_1, it can be shown that the power of such tests remains invariant within a broad class of distributions φ_1. We shall study in detail the case of multiple correlation from which other results follow without much difficulty.

Let the conditional distribution of y for fixed $x_1, x_2, \ldots, x_p$ be given by

$$\varphi_2(y/x) = \frac{1}{\sqrt{(2\pi)}.\sigma}\cdot\exp\left[-\frac{1}{2\sigma^2}\{y-m(x_1, \ldots, x_p)\}^2\right] \tag{13}$$

where

$$E(y/x) = m(x_1, \ldots, x_p) = \alpha+\sum_{i=1}^{p}\beta_i x_i \tag{14}$$

Let us define the population multiple correlation by

$$\rho^2 = \frac{\text{Variation in mean values of } y \text{ due to } x}{\text{Total variation of } y}$$

$$= \frac{E\sum\limits_{i,j}\beta_i\beta_j(x_i-\xi_i)(x_j-\xi_j)}{\sigma^2+E\sum\limits_{i,j}\beta_i\beta_j(x_i-\xi_i)(x_j-\xi_j)} = \frac{\sum\limits_{i,j}\sigma_{ij}\beta_i\beta_j}{\sigma^2+\sum\limits_{i,j}\sigma_{ij}\beta_i\beta_j} \tag{15}$$

where the vector ξ will denote the expectation of the vector x; (σ_{ij}) is the dispersion matrix of x of rank p; $(\sigma_{yy}, \sigma_{y1}, \ldots, \sigma_{yp})$ is the covariance of y and (y, x).

To test the hypothesis $H(\rho = 0)$ which is equivalent to

$$H(\beta_1 = \beta_2 = \ldots = \beta_p = 0) \tag{16}$$

we consider an independent sample of n observations $(y_l, x_{1l}, \ldots, x_{pl})$, $(l = 1, 2, \ldots, n)$. Denote the best linear estimates of β's by b's and put

$$\left.\begin{aligned} S_{ij} &= \sum_{l=1}^{n}(x_{il}-\bar{x}_i)(x_{jl}-\bar{x}_j) \quad i, j = 1, 2, \ldots, p \\ S_{yi} &= \sum_{l=1}^{n}(y_l-\bar{y})(x_{il}-\bar{x}_i) \qquad i = 1, 2, \ldots, p \end{aligned}\right\} \tag{17}$$

Then it is noted that

$$\chi_1^2 = \frac{1}{\sigma^2}\sum S_{ij}b_ib_j, \quad \chi_2^2 = \frac{1}{\sigma^2}\{\sum_l(y_l-\bar{y})^2-\sum S_{ij}b_ib_j\} \tag{18}$$

follow independently χ^2-distribution with p and $n-p-1$ *d.f.* when the hypothesis in (16) is true So Fisher's F-distribution is availed of by taking the ratio of mean squares. We shall, however, consider an equivalent statistic E^2 given by

$$E^2 = \chi_1^2/(\chi_1^2+\chi_2^2). \tag{19}$$

E^2 is, in fact, the sample multiple-correlation coefficient.

The distribution of E^2 when the null hypothesis is true is independent of the distribution of x's, but the power function of E^2 involves the distribution of x's. When we further assume $x_{1t}, x_{2t}, \ldots, x_{pt}$ to come from a non-singular p-variate normal population with dispersion matrix (σ_{ij}), this power function is obtained in the form

$$\int f(E^2)dE^2$$

where

$$f(E^2) = \frac{(1-\rho^2)^{(n-1)/2}}{B(p/2,\ (n-p-1)/2)}.(E^2)^{(p-2)/2}(1-E^2)^{(n-p-3)/2}.{}_2F_1\left(\frac{n-1}{2},\ \frac{n-1}{2},\ \frac{p}{2},\ \rho^2E^2\right). \qquad (20)$$

It will be shewn that *the form* (20) *will remain invariant for all distributions of x such that*

$$E\left[\exp\left\{-\frac{1}{2\sigma^2}\sum_{i,j}S_{ij}\beta_i\beta_j\right\}\right] = \left\{1+\frac{1}{\sigma^2}\sum_{i,j}\sigma_{ij}\beta_i\beta_j\right\}^{-(n-1)/2} \qquad (21)$$

To get this result let us first of all find the conditional power function of E^2 for fixed x's. Since in E^2, χ_1^2 will follow non-central χ^2-distribution with $\lambda = \sum S_{ij}\beta_i\beta_j/\sigma^2$, the distribution of E^2 is immediately obtained as (Tang, 1938)

$$f(E^2/x,\ \lambda) = \frac{e^{-\lambda/2}(E^2)^{(p-2)/2}(1-E^2)^{(n-p-3)/2}}{B(p/2,\ (n-p-1)/2)}.{}_1F_1\left(\frac{n-1}{2},\ \frac{p}{2},\ \frac{\lambda E^2}{2}\right) \qquad (22)$$

To get the unconditional power function of E^2, we have to multiply the expression (22) by the joint distribution of x's and integrate the product over x's, *i.e.*,. we have to evaluate $E\{f(E^2/x,\lambda)\}$. Expanding the right hand side of (22), the expectation of each term is obtained by differentiating both sides of (21) with respect to $1/\sigma^2$ the required number of times and noting the validity of differentiation under integration on the left hand side of (21). This leads to the result in the form (20).

The corresponding result in the case of simple regression will be that the power of the test remains invariant within that class of distribution of the single conditioning variable x for which the mean square follows s^2-distribution of the normal theory.

So far we have considered tests of hypotheses for samples of pre-determined size. If we make use of the *sequential probability ratio* test, it is clear from the form of *p. d.* in (2) that the power function is controlled to a large extent at two points independently of the distribution of the conditioning variables. This is because in taking the probability ratio the distribution φ_1 cancels out and the constants A and B occurring in the sequential test procedure are approximately independent of the random variables (Wald, 1945).

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ON FLEXURE FUNCTIONS OF A SEMI-CIRCULAR CYLINDER

By

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1. In a recent paper (Ghosh, 1947) the problem of pure flexure of an elastic cylinder has been reduced to the determination of four functions f_{11}, f_{21}, f_{12} and f_{22} given by

$$f_{11}+if_{21} = -\frac{i}{4\pi}\int_{\gamma}[\omega(\sigma)]^2\bar{\omega}(1/\sigma)\frac{d\sigma}{\sigma-\zeta}, \tag{1.1}$$

$$f_{11}-if_{21} = \frac{i}{4\pi}\int_{\gamma}\omega(\sigma)[\bar{\omega}(1/\sigma)]^2\frac{d\sigma}{\sigma-\zeta}, \tag{1.2}$$

$$f_{12}+if_{22} = -\frac{i}{2\pi}\int^{\zeta}d\zeta\int_{\gamma}[\omega(\sigma)]^2\bar{\omega}'(1/\sigma)\frac{d\sigma}{\sigma^2(\sigma-\zeta)}, \tag{1.3}$$

$$f_{12}-if_{22} = -\frac{i}{2\pi}\int^{\zeta}d\zeta\int[\bar{\omega}(1/\sigma)]^2\omega'(\sigma)\frac{d\sigma}{\sigma-\zeta}, \tag{1.4}$$

where $\omega(\zeta)$ is the function which maps conformally the cross-section of the cylinder in the z-plane on the unit circle in the ζ-plane, the origin in the z-plane being taken at the centroid of the cross-section.

These canonical flexure functions have been determined in a previous paper (Mitra, 1948) when the cross-section is (1) an inverse of an ellipse with respect to its centre, (2) two orthogonal circular arcs of equal radii, and (3) a loop of the lemniscate of Bernoulli. In the present paper these functions are found for the semi-circular cross-section.

2. The semi-circular area bounded by the arc of the circle $x^2+y^2=1$, $y\geq 0$ and the diameter $y=0\,(-1\leq x\leq 1)$ is mapped conformally on the unit circle in the ζ-plane by the transformation (Sokolnikoff, p. 181)

$$\zeta = \frac{(z+1)^2-i(z-1)^2}{(z+1)^2+i(z-1)^2}$$

or

$$z = \omega(\zeta) = \frac{1+i\zeta-\sqrt{[2(1-\zeta^2)]}}{i+\zeta} \tag{2.1}$$

where in the z-plane, the centre of the semi-circle is the origin.

If the origin in the z-plane be taken at the centroid of the cross-section the transformation formula is

$$z = \omega(\zeta) - ia = \frac{1+i\zeta-\sqrt{[2(1-\zeta^2)]}}{i+\zeta} - ia \tag{2.2}$$

where

$$a = 4/3\pi. \tag{2.3}$$

With this transformation formula (2.2), the formulæ (1.1), (1.2), (1.3) and (1.4) become,

$$f_{11}+if_{21} = -\frac{i}{4\pi}\Big[\int_\gamma \{\omega(\sigma)\}^2\bar{\omega}(1/\sigma)\frac{d\sigma}{\sigma-\zeta} - 2ia\int_\gamma \omega(\sigma)\bar{\omega}(1/\sigma)\frac{d\sigma}{\sigma-\zeta} + ia\int_\gamma \{\omega(\sigma)\}^2\frac{d\sigma}{\sigma-\zeta}$$

$$-a^2\int_\gamma \bar{\omega}(1/\sigma)\frac{d\sigma}{\sigma-\zeta} + 2a^2\int_\gamma \omega(\sigma)\frac{d\sigma}{\sigma-\zeta} - ia^3\int_\gamma \frac{d\sigma}{\sigma-\zeta}\Big], \tag{2.4}$$

$$f_{11}-if_{21} = \frac{i}{4\pi}\Big[\int_\gamma \omega(\sigma)\{\bar{\omega}(1/\sigma)\}^2\frac{d\sigma}{\sigma-\zeta} + 2ia\int_\gamma \omega(\sigma)\bar{\omega}(1/\sigma)\frac{d\sigma}{\sigma-\zeta} - ia\int_\gamma \{\bar{\omega}(1/\sigma)\}^2\frac{d\sigma}{\sigma-\zeta}$$

$$+2a^2\int_\gamma \bar{\omega}(1/\sigma)\frac{d\sigma}{\sigma-\zeta} - a^2\int_\gamma \omega(\sigma)\frac{d\sigma}{\sigma-\zeta} + ia^3\int_\gamma \frac{d\sigma}{\sigma-\zeta}\Big], \tag{2.5}$$

$$f_{12}+if_{22} = -\frac{i}{2\pi}\int^\zeta d\zeta\Big[\int_\gamma \{\omega(\sigma)\}^2\bar{\omega}'(1/\sigma)\frac{d\sigma}{\sigma^2(\sigma-\zeta)} - 2ai\int_\gamma \omega(\sigma)\bar{\omega}'(1/\sigma)\frac{d\sigma}{\sigma^2(\sigma-\zeta)}$$

$$-a^2\int_\gamma \bar{\omega}'(1/\sigma)\frac{d\sigma}{\sigma^2(\sigma-\zeta)}\Big], \tag{2.6}$$

$$f_{12}-if_{22} = -\frac{i}{2\pi}\int^\zeta d\zeta\Big[\int_\gamma \{\bar{\omega}(1/\sigma)\}^2\omega'(\sigma)\frac{d\sigma}{\sigma-\zeta} + 2ai\int_\gamma \bar{\omega}(1/\sigma)\,\omega'(\sigma)\frac{d\sigma}{\sigma-\zeta} + a^3\int_\gamma \omega'(\sigma)\frac{d\sigma}{\sigma-\zeta}\Big]. \tag{2.7}$$

Evaluating the integrals as in the previous paper and observing that $\sigma = -i$ is not a singularity of $\omega(\sigma)$ we have,

$$\int_\gamma [\omega(\sigma)]^2\omega(1/\sigma)\frac{d\sigma}{\sigma-\zeta} = 2\pi i[\omega(\zeta)]^2\frac{1+i\zeta}{i+\zeta} + \frac{32i\zeta}{(i+\zeta)^2} - \frac{8(1-\zeta^2)(1+i\zeta)}{(i+\zeta)^3}\log\frac{i(1+\zeta)}{1-\zeta}, \tag{2.8}$$

$$\int_\gamma \omega(\sigma)\bar{\omega}(1/\sigma)\frac{d\sigma}{\sigma-\zeta} = 2\pi i\omega(\zeta)\frac{1+i\zeta}{i+\zeta} + \frac{8}{i+\zeta} + \frac{4\pi\zeta}{(i+\zeta)^2} - \frac{4(1-\zeta^2)}{(i+\zeta)^2}\log\frac{1+\zeta}{1-\zeta} \tag{2.9}$$

$$\int_\gamma [\omega(\sigma)]^2\frac{d\sigma}{\sigma-\zeta} = 2\pi i[\omega(\zeta)]^2, \tag{2.10}$$

$$\int_\gamma \bar{\omega}(1/\sigma)\frac{d\sigma}{\sigma-\zeta} = \text{constant}, \tag{2.11}$$

$$\int_\gamma \omega(\sigma)\frac{d\sigma}{\sigma-\zeta} = 2\pi i\omega(\zeta), \tag{2.12}$$

$$\int_\gamma \omega(\sigma)[\bar{\omega}(1/\sigma)]^2\frac{d\sigma}{\sigma-\zeta} = -\frac{2\pi i\sqrt{[2(1-\zeta^2)]}}{(i+\zeta)^3}\{(1+i\zeta)^2+2(1-\zeta^2)\}+4\pi i\frac{(1-\zeta^2)(1+i\zeta)}{(i+\zeta)^3}$$

$$+\frac{32i\zeta}{(i+\zeta)^2}-\frac{8(1-\zeta^2)(1+i\zeta)}{(i+\zeta)^3}\log\frac{1+\zeta}{1-\zeta}, \qquad (2.13)$$

$$\int_\gamma \{\omega(1/\sigma)]^2\frac{d\sigma}{\sigma-\zeta} = \text{constant}, \qquad (2.14)$$

$$\int_\gamma [\omega(\sigma)]^2\bar{\omega}'(1/\sigma)\frac{d\sigma}{\sigma^2(\sigma-\zeta)} = \frac{4\pi i[\omega(\zeta)]^2}{(i+\zeta)^2}-\frac{16}{3(i+\zeta)}+\frac{16\zeta}{(i+\zeta)^2}+\frac{8(1+i\zeta)^2}{(i+\zeta)^4}$$

$$-\frac{8(1+i\zeta)^2}{(i+\zeta)^4}\log\frac{i(1+\zeta)}{1-\zeta}, \qquad (2.15)$$

$$\int_\gamma \omega'\sigma)\bar{\omega}'(1/\sigma)\frac{d\sigma}{\sigma^2(\sigma-\zeta)} = \frac{4\pi i\omega(\zeta)}{(i+\zeta)^2}-\frac{\pi i}{i+\zeta}+\frac{8}{(i+\zeta)^2}-\frac{4(1+i\zeta)}{(i+\zeta)^3}\log\frac{i(1+\zeta)}{1-\zeta}, \qquad (2.16)$$

$$\int_\gamma \bar{\omega}'(1/\sigma)\frac{d\sigma}{\sigma^2(\sigma-\zeta)} = 0, \qquad (2.17)$$

$$\int_\gamma [\bar{\omega}(1/\sigma)]^2\omega'(\sigma)\frac{d\sigma}{\sigma-\zeta} = \frac{2\pi i\sqrt{2}\{(1+i\zeta)^2+2(1-\zeta^2)\}(1+i\zeta)}{(i+\zeta)^4\sqrt{(1-\zeta^2)}}+\frac{16}{3(i+\zeta)}-\frac{16\zeta}{(i+\zeta)^2}$$

$$-\frac{8(1+i\zeta)^2}{(i+\zeta)^3}+\frac{8(1+i\zeta)^3}{(i+\zeta)^4}\log\frac{i(1+\zeta)}{1-\zeta}-8\pi i\frac{(1+i\zeta)^2}{(i+\zeta)^4}, \qquad (2.18)$$

$$\int_\gamma \bar{\omega}(1/\sigma)\omega'(\sigma)\frac{d\sigma}{\sigma-\zeta} = \frac{2\pi i\sqrt{2}(1+i\zeta)^2}{(i+\zeta)^3\sqrt{(1-\zeta^2)}}-\frac{\pi i}{i+\zeta}-\frac{8}{(i+\zeta)^2}+\frac{4(1+i\zeta)}{(i+\zeta)^3}\log\frac{i(1+\zeta)}{1-\zeta}-\frac{4\pi i(1+i\zeta)}{(i+\zeta)^3}, \qquad (2.19)$$

$$\int_\gamma \omega'(\sigma)\frac{d\sigma}{\sigma-\zeta} = 2\pi i\omega'(\zeta). \qquad (2.20)$$

Substituting in (2.4), (2.5), (2.6) and (2.7), and using (2.1) and (2.3) we get,

$$f_{11} = \frac{z}{4}-\frac{i}{3\pi}z^2-\frac{4}{3\pi^2 z}-\frac{2}{3\pi^2}\frac{(z^2-1)^2}{z^2}\log\frac{1-z}{1+z}, \qquad (2.21)$$

$$f_{21} = -\frac{i}{4}z^3+\frac{1}{3\pi}z^2-\frac{4iz}{9\pi^2}-\frac{1}{2\pi}\left(z^2+\frac{1}{z^2}\right)+\frac{1}{4\pi}\frac{(z^2+1)(z^2-1)^2}{z^3}\log\frac{1+z}{1-z}, \qquad (2.22)$$

$$f_{12} = \frac{2iz^2}{3\pi}+z\left(\frac{4}{9\pi^2}+\frac{1}{2}\right)+\frac{4}{3\pi^2 z}+\frac{2}{3\pi^2}\frac{(z^2-1)^2}{z^2}\log\frac{1-z}{1+z}, \qquad (2.23)$$

$$f_{22} = \frac{1}{6}iz^3-\frac{8}{9\pi^2}iz+\frac{1}{3\pi}\left(z^2+\frac{1}{z^2}\right)-\frac{(z^2+1)(z^2-1)^2}{4\pi z^3}\log\frac{1+z}{1-z}$$

$$+\frac{1}{12\pi}\left[-32\log(1+z)+\frac{z^6-9z^4+16z^3-9z^2+1}{z^3}\log\frac{1+z}{1-z}\right]. \qquad (2.24)$$

These results can be identified with those given by Ghosh (1948).

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STRESSES IN SOME ÆOLOTROPIC AND ISOTROPIC CIRCULAR DISKS OF VARYING THICKNESS ROTATING ABOUT THE CENTRAL AXIS

By

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Introduction

The problems on the mean values of the stresses in thin rotating circular disks of varying thickness of isotropic material, where the thickness is a function of the distance r from the centre of the disk, have been discussed by Martin (1923), Hodkinson (1923), Malkin (1934), Sen (1935), Bisshopp (1944) and others. Recently stresses in the case of cylindrically æolotropic rotating thin disks of uniform thickness have been determined by Carrier (1943). A cylindrically æolotropic material has been defined by the author as one whose elastic constants as referred to cylindrical coordinates (r, θ, z) are independent of the magnitude of r and remain invariant under the following coordinate transformations: A rotation about the z-axis (which coincides with the axis of the disk), a translation along the z-axis and a reversal of the z-axis.

In the present paper the mean radial and hoop stresses in thin rotating circular disks of *cylindrically aeolotropic* material are determined, where the thickness '$2h$' at a distance r from the axis are given by the relations,

$$h = h_0 r^{-\lambda} \tag{i}$$

and

$$h = h_0 r^{\frac{1-\alpha}{\beta-1}} \exp\left(-\lambda r^{1-\beta}\right) \tag{ii}$$

in which h_0 and λ are constants and α, β two other constants whose values are given later on. It was thought worth-while to consider the profile of the latter type, because this is of the similar form though not exactly the same as that obtained for disks with uniform stress (§ 3).

Finally, as the profile for isotropic disks with uniform stress and varying thickness is of the form $e^{-\lambda r^2}$, it appears that this case deserves some consideration and hence this problem has also been discussed in this paper. It may be noted that Malkin (1934) has treated problems of isotropic disks with thickness varying as $\exp(-\beta r^{1+\nu})$ where $\nu = \pm\frac{1}{2}$.

1. Statement of the problem for cylindrically aeolotropic plates

Assuming that the stresses and displacements do not vary across the thickness of the plate (along the z-axis) and that the lateral surfaces of the plates are free from stress, the analytical expressions of Hooke's Law for the general *aeolotropy* given in

Love (1920) reduce in case of cylindrical *aeolotropy* to the form,

$$e_r = a_{12}\sigma_\theta + a_{22}\sigma_r, \quad e_\theta = a_{11}\sigma_\theta + a_{12}\sigma_r,$$
$$\gamma_{r\theta} = a_{66}\tau_{r\theta}, \quad e_z = a_{13}\sigma_\theta + a_{23}\sigma_r, \tag{1.1}$$

where e_r, e_θ, e_z are respectively the radial, tangential and longitudinal strains, $\gamma_{r\theta}$ the shear strain, σ_r, σ_θ, $\tau_{r\theta}$ the radial, tangential and shear stresses and a_{ij} $(i,j = 1, 2, \ldots 6)$ the elastic constants.

Also, since the relations between strain and displacements are independent of the material, so as in the case of isotropy we have

$$e_r = \frac{du}{dr}, \quad e_\theta = \frac{u}{r} + \frac{1}{r}\frac{dv}{d\theta},$$
$$e_z = \frac{dw}{dz}, \quad \gamma_{r\theta} = \frac{dv}{dr} - \frac{v}{r} + \frac{1}{r}\frac{du}{d\theta}. \tag{1.2}$$

where u, v, w are respectively radial, tangential and longitudinal displacements.

Now, noting that in case of rotating disks, from symmetry $v = 0$, $w =$ constant and u is independent of θ, relations (1.2) reduce to

$$e_r = \frac{du}{dr}, \quad e_\theta = \frac{u}{r}. \tag{1.3}$$

Let P and Q denote the mean radial and hoop stresses and $2h$ be the thickness at r, u the mean radial displacement across the thickness of the disk, ω, the angular velocity and ρ, the mass per unit volume of the material of the disk.

Since, the equilibrium or motion is independent of the type of material, so as in the case of isotropy the equation of motion is given by (Prescott, 1946)

$$hQ - \frac{d}{dr}(hrP) = \rho hr^2\omega^2. \tag{1.4}$$

Now, assuming that there is the same relations between mean stresses and mean strains as between actual stresses and actual strains, we get, from equations (1.1) and (1.3),

$$\frac{du}{dr} = a_{12}Q + a_{22}P \tag{1.41}$$

and

$$\frac{u}{r} = a_{11}Q + a_{12}P. \tag{1.42}$$

Eliminating u from these equations, we get,

$$a_{12}Q + a_{22}P = \frac{d}{dr}[a_{11}rQ + a_{12}rP]. \tag{1.43}$$

Again, eliminating Q from equations (1.4) and (1.43) and writing y for hrP, we get,

$$\frac{d^2y}{dr^2} + \left(\frac{1}{r} - \frac{1}{h}\frac{dh}{dr}\right)\frac{dy}{dr} - \left(\frac{\alpha}{r^2} + \frac{\beta}{rh}\frac{dh}{dr}\right)y = -(3-\beta)\rho\omega^2 rh \tag{1.44}$$

where $\alpha = a_{22}/a_{11}$ and $\beta = a_{12}/a_{11}$.

From equations (1.41) and (1.42), we have

$$P = \frac{1}{a_{11}a_{22}-a_{12}^2}\left[a_{11}\frac{du}{dr}-a_{12}\frac{u}{r}\right] = k_1\frac{du}{dr}-k_2\frac{u}{r} \tag{1.51}$$

$$Q = \frac{1}{a_{11}a_{22}-a_{12}^2}\left[a_{22}\frac{u}{r}-a_{12}\frac{du}{dr}\right] = k_3\frac{u}{r}-k_2\frac{du}{dr} \tag{1.52}$$

where $k_1 = a_{11}/c$, $k_2 = a_{12}/c$, $k_3 = a_{22}/c$, and $c = a_{11}a_{22}-a_{12}^2$.

Putting these values of P and Q in equation (1.4) we get

$$\frac{d^2u}{dr^2}+\left(\frac{1}{r}+\frac{1}{h}\frac{dh}{dr}\right)\frac{du}{dr}-\left(\frac{\alpha}{r^2}+\frac{\beta}{rh}\frac{dh}{dr}\right)u+\frac{\rho r\omega^2}{k_1} = 0. \tag{1.53}$$

The elimination of Q from (1.4) and (1.43) gives

$$a_{12}\left[\frac{1}{h}\frac{d}{dr}(hrP)+\rho r^2\omega^2\right]+a_{22}P = \frac{d}{dr}\left[a_{11}\left\{\frac{r}{h}\frac{d}{dr}(hrP)+\rho r^3\omega^2\right\}+a_{12}rP\right]$$

which reduces to

$$r^2\frac{d^2P}{dr^2}+r\left(3+\frac{r}{h}\frac{dh}{dr}\right)\frac{dP}{dr}+\left\{(1-\alpha)+(2-\beta)\frac{r}{h}\frac{dh}{dr}\right.$$
$$\left.+r^2\frac{d}{dr}\left(\frac{1}{h}\frac{dh}{dr}\right)\right\}P+(3-\beta)\rho r^2\omega^2 = 0. \tag{1.6}$$

For isotropy putting $\alpha = 1$, $\beta = -\nu$ (Poisson's ratio) in equations (1.44), (1.53) and (1.6) the corresponding equations (12.120) of Prescott (1946), equation (37) of Malkin (1934) and of A. Stodola (1927) respectively are obtained.

2. STRESSES IN ROTATING CYLINDRICALLY AEOLOTROPIC PLATES WITH THICKNESS VARYING AS $r^{-\lambda}$

Let

$$h = h_0r^{-\lambda}.$$

Putting $h = h_0r^{-\lambda}$ in equation (1.44) and writing k for $(3-\beta)\rho h_0\omega^2$, we get

$$r^2\frac{d^2y}{dr^2}+(1+\lambda)r\frac{dy}{dr}-(\alpha-\lambda\beta)y = -kr^{3-\lambda}. \tag{2.1}$$

As a particular integral of the equation (2.1) we assume

$$y = Ar^{3-\lambda}.$$

We find that the equation is satisfied if

$$A = \frac{-k}{9-\alpha-\lambda(3-\beta)}.$$

The complementary function is $Br^{\eta_1}+Cr^{\eta_2}$ where B and C are constants of integration to be determined from the boundary conditions and η_1, η_2 $(\eta_1 > \eta_2)$ are the roots of the equation

$$\eta^2+\lambda\eta-(\alpha-\lambda\beta) = 0.$$

Now,

$$P = \frac{y}{rh} = \frac{yr^{\lambda-1}}{h_0} = \frac{1}{h_0}\left[Br^{\eta_1+\lambda-1}+Cr^{\eta_2+\lambda-1}+Ar^2\right]. \tag{2.21}$$

From equations (1.4) and (2.21)

$$Q = \frac{1}{h}\frac{dy}{dr}+\rho r^2\omega^2 = \frac{1}{h_0}\left[B\eta_1 r^{\eta_1+\lambda-1}+C\eta_2 r^{\eta_2+\lambda-1}+A(3-\lambda)r^2\right]+\rho r^2\omega^2. \tag{2.22}$$

The boundary conditions (say) $P = 0$ at $r = a$ and $r = b$ $(a > b)$, a being the radius of the disk and b that of the bore at the centre of the disk, give

$$B = Aa^2\frac{1-l^{3-\eta_2-\lambda}}{1-l^{\eta_1-\eta_2}}\cdot\frac{1}{a^{\eta_1+\lambda+1}}, \quad \text{and} \quad C = Aa^2\frac{1-l^{3-\eta_1-\lambda}}{1-l^{\eta_2-\eta_1}}\cdot\frac{1}{a^{\eta_2+\lambda-1}}.$$

where $l = b/a$.

Therefore,

$$P = \frac{\rho\omega^2a^2(3-\beta)}{\alpha-9+\lambda(3-\beta)}\left[\left(\frac{r}{a}\right)^2-\frac{1-l^{3-\eta_1-\lambda}}{1-l^{\eta_2-\eta_1}}\left(\frac{r}{a}\right)^{\eta_2+\lambda-1}-\frac{1-l^{3-\eta_2-\lambda}}{1-l^{\eta_1-\eta_2}}\left(\frac{r}{a}\right)^{\eta_1+\lambda-1}\right] \tag{2.31}$$

and

$$Q = \frac{\rho\omega^2a^2(3-\beta)}{\alpha-9+\lambda(3-\beta)}\left[\left\{3+\frac{\alpha-9}{3-\beta}\right\}\left(\frac{r}{a}\right)^2-\eta_2\frac{1-l^{3-\eta_1-\lambda}}{1-l^{\eta_2-\eta_1}}\left(\frac{r}{a}\right)^{\eta_2+\lambda-1}\right.$$

$$\left.-\eta_1\frac{1-l^{3-\eta_2-\lambda}}{1-l^{\eta_1-\eta_2}}\left(\frac{r}{a}\right)^{\eta_1+\lambda-1}\right]. \tag{2.32}$$

For plates of uniform thickness of the same aeolotropic material, $\lambda = 0$.

Putting $\lambda = 0$ in (2.31) and (2.32) the results given by Carrier (1943) are obtained. Again for isotropy, $\beta = -\nu$ and $\alpha = 1$. Putting these values of α and β in (2.31) and (2.32), the results given in equations (12.127) and (12.128) of Prescott (1946) are obtained; c, b and l being zero for a complete disk.

3. ROTATING CYLINDRICALLY AEOLOTROPIC DISK OF VARYING THICKNESS WITH UNIFORM STRESS

Consider the thickness $2h$ of the disk to be an unknown quantity to be determined from the conditions for which the stresses P and Q are equal. Now putting $P = Q = S$ in the equations (1.41) and (1.42) we get,

$$\alpha_1\frac{du}{dr} = \alpha_2\frac{u}{r} \tag{3.1}$$

where $\alpha_1 = a_{11}+a_{12}$, and $\alpha_2 = a_{22}+a_{12}$, whence

$$u = A_0 r^{\alpha_2/\alpha_1} \tag{3.2}$$

A_0 being a constant.

From (1.42) and (3.2)

$$P = Q = \frac{1}{\alpha_1}\frac{u}{r} = \frac{A_0}{\alpha_1} r^{\alpha_2/\alpha_1 - 1}. \tag{3.3}$$

Now, from (1.4) and (3.3) we get, writing p' for $1-\alpha_2/\alpha_1$

$$hr^{-p'} - \frac{d}{dr}(hr^{1-p'}) = \frac{\alpha_1}{A_0}\rho h r^2 \omega^2$$

or

$$\frac{1}{h}\frac{dh}{dr} = \frac{p'}{r} - \frac{\alpha_1}{A_0}\rho\omega^2 r^{1+p'}$$

whence,

$$h = B_0 r^{p'} \exp(-\lambda_0 r^{q'}) \tag{3.4}$$

where B_0 is a constant,

$$\lambda_0 = \frac{\alpha_1 \rho\omega^2}{A_0(2+p')} \quad \text{and} \quad q' = 2+p'.$$

For isotropic material $\alpha_1 = \alpha_2$; hence $p' = 0$, $q' = 2$ and $A_0 = S$.

Thus, in the case of isotropy equation (3.4) becomes

$$h = B_0 \exp\left(-\frac{\rho\omega^2}{2S} r^2\right) \tag{3.5}$$

which is the same as equation (12.131) of Prescott (1946).

4. Rotating cylindrically aeolotropic disk of thickness $2h$ given by the relation $h = h_0 r^p \exp(-\lambda r^q)$.

Let $h = h_0 r^p \exp(-\lambda r^q)$ [*cf.*, result (3.4)]. For simplicity take $p = (1-\alpha)/(\beta-1)$ and $q = 1-\beta$ so that the equation to the profile becomes

$$h = h_0 r^{\frac{1-\alpha}{\beta-1}} \exp(-\lambda r^{1-\beta}). \tag{4}$$

Putting $h = h_0 r^p \exp(-\lambda r^q)$ in equation (1.6), we get

$$r^2\frac{d^2P}{dr^2} + (3+p-\lambda q r^q) r\frac{dP}{dr} - 2\lambda q^2 r^q P + (3-\beta) r^2 \rho\omega^2 = 0. \tag{4.1}$$

The complementary function is obtained from the equation

$$r^2\frac{d^2P}{dr^2} + (3+p-\lambda q r^q) r\frac{dP}{dr} - 2\lambda q^2 r^q P = 0. \tag{4.2}$$

Putting $\zeta = \lambda r^q$ the equation (4.2) reduces to

$$\zeta\frac{d^2P}{d\zeta^2} + (m-\zeta)\frac{dP}{d\zeta} - 2P = 0, \tag{4.21}$$

where $m = 1+(p+2)/q$. The solution of this well-known equation in terms of confluent hypergeometric function is

$$P = A_1 M(2, m, \zeta) + B_1 \zeta^{1-m} M(3-m, 2-m, \zeta), \tag{4.22}$$

where A_1 and B_1 are constants of integration and $M(2, m, \zeta)$, $M(3-m, 2-m, \zeta)$ stand respectively for the absolutely and uniformly convergent series

$$1+\frac{2}{m}\zeta+\frac{2.3}{m(m+1)}\frac{\zeta^2}{2\,!}+\frac{2.3.4.}{m(m+1)(m+2)}\frac{\zeta^3}{3\,!}+\cdots \tag{4.31}$$

and

$$1+\frac{3-m}{2-m}\zeta+\frac{(3-m)(4-m)}{(2-m)(3-m)}\frac{\zeta^2}{2\,!}+\frac{(3-m)(4-m)(5-m)}{(2-m)(3-m)(4-m)}\frac{\zeta^3}{3\,!}+\cdots=\left(1+\frac{\zeta}{2-m}\right)e^{\zeta} \tag{4.32}$$

For the *aeolotropic* material of the type under consideration, taking $\alpha = 4$, $\beta = -0{\cdot}5$ (Carrier, 1943),

$$p = \frac{1-\alpha}{\beta-1} = 2 \quad \text{and} \quad q = 1-\beta = \tfrac{3}{2}.$$

The equation (4) to the profile reduces to

$$h = h_0 r^3 \exp\left(-\lambda r^{3/2}\right) \tag{4.4}$$

and $m = \frac{11}{3}$.

The equation (4.1) in terms of ζ is

$$\zeta\frac{d^2P}{d\zeta^2}+(m-\zeta)\frac{dP}{d\zeta}-2P = -\frac{(3-\beta)\rho\omega^2}{q^2\lambda^{2/q}}\zeta^{2/q-1}. \tag{4.41}$$

Putting $p = 2$, $q = \frac{3}{2}$, $m = \frac{11}{3}$ this equation becomes

$$\zeta\frac{d^2P}{d\zeta^2}+\left(\frac{11}{3}-\zeta\right)\frac{dP}{d\zeta}-2P = -\frac{14}{9}\frac{\rho\omega^2}{\lambda^{4/3}}\zeta^{\frac{1}{3}}. \tag{4.42}$$

The complementary function of this equation from (4.22), (4.31) and (4.32) is

$$A_1M(2, \tfrac{11}{3}, \zeta)+B_1\zeta^{-8/3}(1-\tfrac{3}{5}\zeta)e^{\zeta}.$$

For the particular integral, we assume

$$P = \zeta^{\frac{1}{3}}\left[A_2+\frac{B_2}{\zeta}+\frac{C_2}{\zeta^2}+\frac{D_2}{\zeta^3}\right]. \tag{4.51}$$

We find that the equation (4.42) is satisfied if

$$\begin{aligned} A_2 &= \frac{2}{3}\frac{\rho\omega^2}{\lambda^{4/3}}, & B_2 &= \frac{1}{2}\frac{\rho\omega^2}{\lambda^{4/3}} \\ C_2 &= -2\frac{\rho\omega^2}{\lambda^{4/3}}, & D_2 &= -5\frac{\rho\omega^2}{\lambda^{4/3}}. \end{aligned} \tag{4.52}$$

Therefore,

$$P = A_1M(2, \tfrac{11}{3}, \zeta)+B_1(1-\tfrac{3}{5}\zeta)\frac{e^{\zeta}}{\zeta^{8/3}}+\frac{\rho\omega^2}{3\lambda^{4/3}}\zeta^{\frac{1}{3}}\left[2+\frac{3}{2\zeta}-\frac{6}{\zeta^2}-\frac{15}{\zeta^3}\right]. \tag{4.53}$$

The constants A_1 and B_1 are to be determined from the boundary conditions.

Substitution of this value of P in equation (1.4) gives

$$Q = A_1\left[3 + \frac{6}{11}\cdot\frac{7}{4}\zeta + \frac{6.9}{11.14}\cdot\frac{4}{3}\,\frac{\zeta^2}{2!} + \frac{6.9.12}{11.14.17}\cdot\frac{9}{8}\,\frac{\zeta^3}{3!} + \cdots\right.$$

$$\left. + \frac{6.9.12\ldots(3n+3)}{11.14.17\ldots(3n+8)}\left(3 - \frac{5}{2}\,\frac{n}{n+1}\right)\frac{\zeta^n}{n!} + \cdots\right]$$

$$-B_1(1+\tfrac{3}{10}\zeta)\,\frac{e^\zeta}{\zeta^{8/3}} + \frac{\rho\omega^2}{3\lambda^{4/3}}\zeta^{\frac{1}{3}}\left[\frac{19}{4} + \frac{12}{\zeta} + \frac{39}{2\zeta^2} + \frac{15}{\zeta^3}\right]. \tag{4.54}$$

Now, let $2h_a$ and $2h_b$ be the thickness at the rim and at the bore respectively, so that from (4.4)

$$\frac{h_b}{h_a} = \frac{b^2}{a^2}\exp\{\lambda(a^{3/2} - b^{3/2})\},$$

whence

$$\lambda = \frac{2{\cdot}303}{a^{3/2} - b^{3/2}}\log_{10}\left(\frac{a^2h_b}{b^2h_a}\right). \tag{4.6}$$

Writing

$$\phi_1(\zeta) = (1 - \tfrac{3}{5}\zeta)\,\frac{e^\zeta}{\zeta^{8/3}}, \qquad \phi_2(\zeta) = (1 + \tfrac{3}{10}\zeta)\,\frac{e^\zeta}{\zeta^{8/3}}$$

$$\psi_1(\zeta) = \zeta^{\frac{1}{3}}\left(2 + \frac{3}{2\zeta} - \frac{6}{\zeta^2} - \frac{15}{\zeta^3}\right),$$

$$\psi_2(\zeta) = \zeta^{\frac{1}{3}}\left(\frac{19}{4} + \frac{12}{\zeta} + \frac{39}{2\zeta^2} + \frac{15}{\zeta^3}\right).$$

and

$$f(\zeta) = 3 + \frac{6}{11}\cdot\frac{7}{4}\zeta + \frac{6.9}{11.14}\cdot\frac{4}{3}\,\frac{\zeta^2}{2!} + \frac{6.9.12}{11.14.17}\cdot\frac{9}{8}\,\frac{\zeta^3}{3!} + \cdots$$

we get, from (4.53) and (4.54)

$$P = A_1M(2, \tfrac{11}{3}, \zeta) + B_1\phi_1(\zeta) + A^1\psi_1(\zeta), \tag{4.61}$$

$$Q = A_1f(\zeta) - B_1\phi_2(\zeta) + A^1\psi_2(\zeta), \tag{4.62}$$

where $A^1 = \rho\omega^2/3\lambda^{4/3}$.

From the boundary conditions $P = 0$, $r = b$, at the bore, (the stress at the bore being a comparatively small amount) and $P = 0$, $r = a$ (at the rim) we get

$$A_1M_a + B_1\phi_1(\zeta_a) + A^1\psi_1(\zeta_a) = 0 \tag{4.63}$$

and

$$A_1M_b + B_1\phi_1(\zeta_b) + A^1\psi_1(\zeta_b) = 0 \tag{4.64}$$

where

$$\begin{aligned} M_a &= [M(2, \tfrac{11}{3}, \zeta)]_{r=a}, \quad \phi_1(\zeta_b) = [\phi_1(\zeta)]_{r=b}, \quad \phi_1(\zeta_a) = [\phi_1(\zeta)]_{r=a}, \\ M_b &= [M(2, \tfrac{11}{3}, \zeta)]_{r=b}, \quad \psi_1(\zeta_b) = [\psi_1(\zeta)]_{r=b}, \quad \psi_1(\zeta_a) = [\psi_1(\zeta)]_{r=a}. \end{aligned} \tag{4.65}$$

From equations (4.63) and (4.64), we get

$$\frac{A_1}{\begin{vmatrix} \phi_1(\zeta_a) & \psi_1(\zeta_a) \\ \phi_1(\zeta_b) & \psi_1(\zeta_b) \end{vmatrix}} = \frac{B_1}{\begin{vmatrix} \psi_1(\zeta_a) & M_a \\ \psi_1(\zeta_b) & M_b \end{vmatrix}} = \frac{A^1}{\begin{vmatrix} M_a & \phi_1(\zeta_a) \\ M_b & \phi_1(\zeta_b) \end{vmatrix}} \tag{4.66}$$

Now, for a particular case, taking $a = 18''$, $b = 6''$, $2h_a = 6''$, $2h_b = 2{\cdot}5''$ we get from equation (4.6)

$$\lambda = 0{\cdot}02143$$

and from the relation $\zeta = \lambda r^{3/2}$

$$\zeta_a = [\zeta]_{r=a} = 1{\cdot}656'',$$
$$\zeta_b = [\zeta]_{r=b} = 0{\cdot}315''. \tag{4.7}$$

Also from the relations (4.65), we have

$$\begin{aligned} \phi_1(\zeta_a) &= 0{\cdot}0254, & \phi_1(\zeta_b) &= 24.19, \\ \psi_1(\zeta_a) &= -3{\cdot}2394, & \psi_1(\zeta_b) &= -363{\cdot}0974, \\ M_a &= 2{\cdot}6158\,; & M_b &= 1{\cdot}1905. \end{aligned} \tag{4.71}$$

Therefore, from (4.66) and (4.71)

$$A_1 = 61.22\rho\omega^2, \quad B_1 = 873.6\rho\omega^2. \tag{4.72}$$

It may be noted that the values of stresses obtained by Malkin (1934) for a rotating *isotropic disk* of varying thickness, may be deduced from the above, by putting $\alpha = 1$, $\beta = -\nu = -\frac{1}{3}$.

Thus, when $\alpha = 1$, $\beta = -\frac{1}{3}$, we have, $p = 0$, $q = \frac{4}{3}$ and $m = \frac{5}{2}$. The equation $h = h_0 r^p \exp(-\lambda r^q)$ of the profile reduces to

$$h = h_0 \exp(-\lambda r^{4/3})$$

which is the 'first exponential profile' considered by Malkin (1934).

Substitution of these values of p, q in (4.41) gives

$$\zeta \frac{d^2P}{d\zeta^2} + (\tfrac{5}{2} - \zeta)\frac{dP}{d\zeta} - 2P = -\frac{15}{8}\frac{\rho\omega^2}{\lambda^{3/2}}\zeta^{\frac{1}{4}}. \tag{4.8}$$

The complementary function of (4.8) is obtained from

$$\zeta \frac{d^2P}{d\zeta^2} + (\tfrac{5}{2} - \zeta)\frac{dP}{d\zeta} - 2P = 0 \tag{4.81}$$

which is the same as equation (5) of Malkin (1934) and its solution from (4.22) and (4.32) is

$$P = C_1 M(2, \tfrac{5}{2}, \zeta) + D_1 \zeta^{-3/2}(1 - 2\zeta)e^{\zeta} \tag{4.82}$$

where

$$M(2, \tfrac{5}{2}, \zeta) = 1 + \tfrac{4}{5}\zeta + \frac{4.6}{5.7}\frac{\zeta^2}{2!} + \frac{4.6.8}{5.7.9}\frac{\zeta^3}{3!} + \ldots$$

and the complementary function obtained is the same as that obtained by Malkin (1934) equation (8).

For the particular integral of (4.8), we assume, as in the case of *aeolotropy*,

$$P = \zeta^{\frac{1}{2}}\left[A_3+\frac{B_3}{\zeta}+\frac{C_3}{\zeta^2}\right].$$

We find the equation (4.8) is satisfied if

$$A_3 = \frac{3}{4}\frac{\rho\omega^2}{\lambda^{3/2}}, \quad B_3 = \frac{1}{2}\frac{\rho\omega^2}{\lambda^{3/2}} \quad \text{and} \quad C_3 = -\frac{1}{2}\frac{\rho\omega^2}{\lambda^{3/2}}. \tag{4.33}$$

Therefore the particular integral is also the same as that obtained by Malkin (1934).

5. STRESSES IN ROTATING ISOTROPIC DISKS WITH THICKNESS VARYING AS $e^{-\lambda r^2}$

In what follows we shall determine mean values of radial and hoop stresses P and Q in rotating disks of isotropic material of varying thickness $2h$ at a distance r from the centre, connected by the relation

$$h = h_0 e^{-\lambda r^2} \tag{5.1}$$

h_0 and λ being constants.

It may be noted that the corresponding problem of bending has been considered by Timoshenko (1940).

In the case of isotropic material equation (1.44) becomes

$$\frac{d^2y}{dr^2}+\left(\frac{1}{r}-\frac{1}{h}\frac{dh}{dr}\right)\frac{dy}{dr}-\left(\frac{1}{r^2}-\frac{\nu}{rh}\frac{dh}{dr}\right)y = -(3+\nu)\rho\omega^2 rh \tag{5.2}$$

ν being Poisson's ratio.

We put $\lambda' = \lambda a^2$, $x = r/a$, a being the radius of the disk, and $y = e^{-\lambda' x^2}R$, where R is a function of x only.

Then, equation (5.1) of the profile becomes

$$h = h_0 e^{-\lambda' x^2}.$$

Also, equation (5.2) gives

$$\frac{d^2R}{dx^2}+\left(\frac{1}{x}-\delta x\right)\frac{dR}{dx}-\left(\frac{1}{x^2}+\delta\sigma\right)R = -kx \tag{5.3}$$

where $\delta = 2\lambda'$, $\sigma = 2+\nu$ and $k = (3+\nu)\rho\omega^2 a^3 h_0$.

It is evident that

$$R_1 = \frac{k}{\delta(1+\sigma)}x = \frac{\rho\omega^2 a^3 h_0}{2\lambda'}x \tag{5.41}$$

is a particular solution of equation (5.3). The complementary function is obtained from,

$$\frac{d^2R}{dx^2}+\left(\frac{1}{x}-\delta x\right)\frac{dR}{dx}-\left(\frac{1}{x^2}+\delta\sigma\right)R = 0. \tag{5.42}$$

Putting,

$$R = \sum_{n=0}^{\infty} E_n x^n,$$

the solution of (5.42) is found to be

$$R = E_0\left[1+\frac{\delta\sigma}{1.3}x^2+\frac{\delta^2\sigma(2+\sigma)}{3.3.5}x^4+\frac{\delta^3\sigma(2+\sigma)(4+\sigma)}{3.3.5.5.7}x^6+\ldots\right]$$

$$+E_1\left[x+\frac{\delta(1+\sigma)}{2.4}x^3+\frac{\delta^2(1+\sigma)(3+\sigma)}{2.4.4.6}x^5+\frac{\delta^3(1+\sigma)(3+\sigma)(5+\sigma)}{2.4.4.6.6.8}x^7+\ldots\right],$$

where E_0 and E_1 are any two constants of integration to be determined from the boundary conditions.

For a complete disk $E_0 = 0$ as otherwise the stress at the centre of the disk, $x = 0$ will be infinitely great.

Hence, the complementary function of (5.3) is the uniformly convergent series

$$E_1\left[x+\sum_{n=1}^{\infty}\frac{\delta^n(1+\sigma)(3+\sigma)\ldots(2n-1+\sigma)}{2.4.4.6.6\ldots 2n.2n.(2n+2)}x^{2n+1}\right] = E_1\phi(x), \text{ (say)}. \tag{5.5}$$

Thus, the solution of equation (5.3) is

$$R = \frac{\rho h_0 a^3\omega^2}{2\lambda'}x+E_1\phi(x). \tag{5.6}$$

Hence we have,

$$P = \frac{y}{rh} = \frac{e^{-\lambda' x^2}R}{axh_0 e^{-\lambda' x^2}} = \frac{R}{ahx_0} = \frac{\rho a^2\omega^2}{2\lambda'}+\frac{E_1}{ah_0}\cdot\frac{1}{x}\phi(x).$$

From the boundary condition, $P = 0$ at $r = a$ or $x = 1$ we get

$$E_1 = -\frac{\rho h_0 a^3\omega^2}{2\lambda'[\phi(x)]_{x=1}}.$$

Therefore,

$$P = \frac{\rho a^2\omega^2}{2\lambda'}-\frac{\rho a^2\omega^3}{2\lambda'[\phi(x)]_{x=1}}\cdot\frac{1}{x}\phi(x). \tag{5.71}$$

Since, $\phi(x)$ is uniformly convergent, it can be differentiated and on substituting this value of P in equation (1.4) Q is determined. Thus,

$$Q = \frac{1}{h}\frac{dy}{dr}+\rho r^2\omega^2 = \frac{1}{ah_0}\frac{dR}{dx}-\frac{2\lambda'}{ah_0}xR+\rho a^2\omega^2x^2$$

$$= \frac{\rho a^2\omega^2}{2\lambda^1}\left[1-\frac{\phi'(x)}{[\psi(x)]_{x=1}}\right]+\rho\omega^2a^2x\frac{\phi(x)}{[\phi(x)]_{x=1}} \tag{5.72}$$

where

$$\phi'(x) = 1+\sum_{n=1}^{\infty}\frac{(2n+1)\delta^n(1+\sigma)(3+\sigma)\ldots(2n-1+\sigma)}{2.4.4.6.6\ldots 2n.2n.(2n+2)}x^{2n}.$$

For a particular case, taking $\lambda' = \frac{1}{2}$, $\delta = 1$, $\nu = 0{\cdot}3$, $\sigma = 2{\cdot}3$, $x = 1$

$$[\phi(x)]_{x=1} = 1+\frac{3{\cdot}3}{2.4}+\frac{3{\cdot}3.5{\cdot}3}{2.4.4.6}+\frac{3{\cdot}3.5{\cdot}3.7{\cdot}3}{2.4.4.6.6.8}+\cdots = 1{\cdot}5192\ldots$$

$$E_1 = -\frac{\rho h_0 a^3\omega^2}{1{\cdot}5192}.$$

Also, taking $\lambda' = \frac{1}{4}$, $\delta = \frac{1}{2}$, $\nu = 0{\cdot}3$, $\sigma = 2{\cdot}3$, $x = 1$

$$E_1 = -\frac{\rho h_0 a^3 \omega^2}{0{\cdot}6154}.$$

In conclusion I am thankful to Dr. B. Sen for his kind help in the preparation of this paper

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SEGRE'S QUARTIC LOCUS*

By

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We first establish *Pappus' theorem in* S_4, space of four dimensions with the help of cross-ratios and hence deduce the existence of a *quadric-point-cone or hyper-cone.* Further we define a *double-four of lines* and thence deduce the existence of an *associated set of five lines.* Again we proceed to show that there can be drawn planes to meet all the five lines: ∞^1 such planes through a point on any one of the five lines and only two such planes in general from a general point not, of course, on any line of the set. Then we discuss the conditions under which the latter two planes may coincide and lastly determine the *Locus of the points through which the planes drawn to meet the lines of an associated set are coincident.* By (2,2) correspondence, discovered in the discussion, this *locus is found to be of the fourth order.*

1. Pappus Theorem

Given two sets of planes through a point O, each set consisting of three planes such that every one belonging to one set meets every plane of the other in a line and no two of a set have a line in common. Let there be a fourth plane of the first set with the property of its members and a similar one belong to the second set; then these latter planes meet in a line.

Let Oa, Ob, Oc, Od be the planes of one set and Oa', Ob', Oc' Od', of the other set such that:—

Oa, Ob, Oc, Od meet Oa' in the lines OA, OB, OC, OD respectively,
Ob' in the lines OA, OB, OC, OD respectively,
Oc' in the lines OA, OB, OC, OD respectively,
and let Oa, Ob, Oc, meet Od' in the lines OA', OB', OC' respectively.

Then we have to prove that Od, Od' meet in a line OD'. Now choose OD' in Od' such that $Ob'(Oa, Ob, Oc, Od) = O(A', B', C', D')$, then OD' must lie in the prime $Ob'd$; further OD' is seen to lie in the prime $Oc'd$ also, therefore $O(A', B', C', D') = Ob'(Oa, Ob, Oc, Od) = O(A, B, C, D) = Oc'(Oa, Ob, Oc, Od)$. Or OD' lies in the common plane of the primes $Ob'd$, $Oc'd$, *i.e.*, Od. Hence the result.

Note. In the above proof we use the fact that the Cross-ratio of a pencil of 4 primes is equal to the cross-ratio of the pencil of 4 lines obtained from them by a plane section and vice-versa.

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* Baker, H. F. Principles of Geometry, Vol. 4, Chap. 5.

HYPER-CONE

Here we define then a *quadric-point cone* or *hyper-cone* as generated by two systems of planes through a point, called the *vertex* of the cone such that planes of a system meet one another in the vertex only and every one belonging to one system meets everyone of the other in a line, which evidently passes through the vertex.

2. ASSOCIATED LINES

Let a, b, c, d; a', b', c', d' be two sets of four lines each forming a *double-four* such that a' meets b, c, d; a meets b', c', d'; The four primes aa', bb', cc', dd' are easily seen to have a line common. Let us call it e. Then the two sets of lines a, b, c, d, e and a', b', c', d', e are such that the planes meeting any four lines of a set meet the fifth also. Such a set of five lines is called an *Associated Set.*

Let O be a point on e and consider the plane Oa' which meets the planes Ob, Oc, Od in lines, for a' intersects b, c, d; Oa' also meets the plane Oa in a line, for these are planes in the same prime aa', O being a point on a line in it. Similarly we can prove that Ob', Oc', Od' meet all the planes Oa, Ob, Oc, Od in lines. Hence these planes generate a hyper-cone with its vertex at O or there exists ∞^1 planes through O which meet the planes Oa, Ob, Oc, Od in lines, and therefore meet the lines a, b, c, d and also e, for O is a point on it. Similarly there exist ∞^1 planes through O that meet a', b', c', d', e.

From a general point O, not on any line of a set, we can draw generally two planes only which intersect all the lines of the set.

3. COINCIDENT PLANES

Let Op, Oq be the two planes through a general point O meeting lines a, b, c, d, e and Op', Oq' meeting a', b', c', d', e. It can be shown that the point where Op, Oq intersect e are the same as those of its intersection with the planes Op', Oq'. Let us call these points U, T such that OU is the common line of Op, Op' and OT that of Oq, Oq'.

Consider the section by Op of the hyper-cone with its vertex at T generated by planes of the type Oq, Oq'. Now planes of the system Oq can be drawn through the points A, B, C, D where Op intersects a, b, c, d. Hence the section is the conic $OABCD$, for O is a point on the hyper-cone under consideration as well as on the plane Op. Evidently this conic cannot pass through U, unless T comes into coincidence with it, *i.e.*, it happens only when Op, Oq coincide in which case Op', Oq' coincide also. Or let a plane Op intersect the lines a, b, c, d, e in the points A, B, C, D, U respectively and take a point O on the conic determined by $ABCDU$ other than these. Let the other plane which we should be able to draw through O cutting the lines a, b, c, d, e be say Oq intersecting e in T, then the section by Op of the hyper-cone with vertex at T generated by planes intersecting a,b,c,d is the conic $OABCD$ as shown above but this is the same as $ABCDU$, hence T must coincide with U, that is possible only if Op coincide with Oq.

All that we have proved above is that; through the points of the conic determined by the intersections of the lines of an associated set with a plane meeting them we can draw either ∞^1 planes that meet the lines or none except this one; conversely, all points, in this plane, through which the other planes meeting the lines of the set are either ∞^1 or coincide with it, lie on a conic, *i.e., the locus of a point, in a plane intersecting the lines of an associated set such that either* ∞^1 *planes meeting the lines pass through it or none except the given one, is the conic determined by its intersections with the lines.*

4. QUARTIC LOCUS

Let P be a point on an arbitrary line p and let the planes through P meeting a, b, c, d, e cut e in R, R'; then the hyper-cones with vertices R and R' generated by planes intersecting a, b, c, d and a', b', c', d' intersect p in the points say Q, Q' other than P. We may say then that to P on p correspond two points Q, Q' on it. Similarly to Q will correspond two points P, P'. Hence the points (P) and (Q) on an arbitrary line p are in $(2, 2)$ correspondence.

Therefore there are four points on p, the double elements of the $(2, 2)$ correspondence on it, such that the two planes drawn through each of them cutting a, b, c, d, e coincide, as well as those cutting a', b', c', d', e' *Thus the locus of such points is a Quartic, whose section by a plane meeting lines of the associated set is the quartic curve consisting of the conic through its intersections with the lines of the set counted twice over, i.e., a double conic.*

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ON ABEL-CONVERGENT INTEGRALS AND THEIR APPLICATION TO MATHEMATICAL PHYSICS

By

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Introduction

This paper deals with certain properties of infinite integrals which oscillate in the ordinary sense. The properties are analogous to those treated in the author's previous papers (Ghosh, 1947, 1948, *Cf.* also Hardy, 1949, p. 349) With the help of these integrals Rutherford scattering formula has been treated by Born's method and the result has been obtained without any special hypothesis unlike Wentzel (1927). The evaluation of a very important infinite integral used by Bethe (1930) has also been indicated.

Mathematical Theory

It is well known that if a function $a(x)$ integrable-R tends to l as x tends to infinity, then

$$\frac{1}{x}\int_0^x a(t)dt$$

also tends to the same limit l as x tends to infinity. If $a(x)$ diverges to infinity, then

$$\frac{1}{x}\int_0^x a(t)dt$$

also diverges to infinity. Again, if

$$\frac{1}{x}\int_0^\infty a(t)dt$$

tends to l as x tends to infinity, then

$$\frac{1}{x}\int_0^\infty e^{-t/x}a(t)dt$$

also tends to l as x tends to infinity. Moreover

$$\frac{1}{x}\int_0^\infty e^{-t/x}a(t)dt$$

diverges to infinity if

$$\frac{1}{x}\int_0^x a(t)dt$$

diverges to infinity as x tends to infinity. Thus we see that if $a(x)$ tends to a limit (finite or not)

$$\frac{1}{x}\int_0^\infty e^{-t/x}a(t)dt$$

tends to the same limit as x tends to infinity.

Definition. *If*

$$\frac{1}{x}\int_0^\infty e^{-t/x}a(t)dt$$

tends to a limit l *as* $x \to \infty$, *we call this limit the Abel limit of* $a(x)$ *as* $x \to \infty$. We denote this by

$$(A)\text{-}\lim_{x\to\infty} a(x) = l.$$

Example: In the ordinary sense $\sin x$ oscillates as $x \to \infty$, but the Abel limit of $\sin x$ as $x \to \infty$, *viz.*,

$$(A)\text{-}\lim_{x\to\infty} \sin x = \lim_{x\to\infty} \frac{1}{x}\int_0^\infty e^{-t/x}\sin t\, dt = 0.$$

In case the function $a(x)$ is not defined, say, from 0 to b, we consider another function $A(x)$, such that

$$A(x) = 0, \qquad 0 \leq x \leq b,$$
$$A(x) = a(x), \quad x > b,$$

and take the limit of

$$\frac{1}{x}\int_0^\infty e^{-t/x}A(t)dt$$

(provided the limit exists) as $x \to \infty$ as the (A)-limit of $a(x)$ as $x \to \infty$.

Let us now consider the integral

$$F(x) = \int_a^x f(t)dt. \tag{1}$$

If (A)-lim $F(x)$ exists as $x \to \infty$, we say that the integral on the right of (1) is (A)-convergent and we take this limit as the value of the infinite integral. We denote this integral by

$$(A)\text{-}\int_a^\infty f(x)dx.$$

Example: Let us consider the integral

$$F(x) = \int_0^\infty \sin t\, dt = 1 - \cos x$$

In the ordinary sense $F(x)$ oscillates as $x \to \infty$, but

$$(A)\text{-}\int_0^\infty \sin x\,dx = 1.$$

In the case of $c = 0$, the integral takes an elegant form :

$$(A)\text{-}\int_0^\infty f(x)dx = \lim_{x\to\infty} \frac{1}{x} \int_0^\infty e^{-t/x}dt \int_0^t f(z)dz = \lim_{x\to\infty} \int_0^\infty e^{-t/x}f(t)dt$$

provided that

$$e^{-t/x}\int_0^t f(z)dz \to 0 \text{ as } t\to\infty, \quad (x>0).$$

In this form or in its equivalent form, *viz.*,

$$\lim_{k\to 0} \int_0^\infty e^{-kt}f(t)dt,$$

it is customary to call the integral Abel summable. We are, however, dealing with a slightly more general notion and we use the word 'convergence' instead of 'summable'.

We would like to mention here that Abel convergence of an integral does not necessarily imply its Cesàro convergence and vice versa, not to speak of convergence to the same limit. It is under certain restrictions that they are the same. (*Cf.* Hardy, 1949, Art. 6.8).

UNIFORM CONVERGENCE

Let us consider integrals containing a parameter. *Let*

$$F(\lambda, a) = \frac{1}{\lambda} \int_0^\infty e^{-x/\lambda}dx \int_c^x f(t, a)dt. \tag{2}$$

If now $F(\lambda, a)$ *converges uniformly in an interval* $A \leq a \leq B$ *as* $\lambda \to \infty$, *we say that the integral*

$$(A)\text{-}\int_c^\infty f(t, a)dt \tag{3}$$

converges uniformly in the interval $A \leq a \leq B$.

We now proceed to study some important properties of (A)-convergent integrals containing a parameter.

CONTINUITY WITH RESPECT TO A PARAMETER

Theorem 1. *Let*

(i) $f(x, a)$ *be continuous in the region* $c \leq x \leq \infty$, $A \leq a \leq b$;

$$\text{(ii)} \quad \frac{1}{\lambda} \int_0^\infty e^{-x/\lambda}dx \int_c^x f(t, a)dt, \quad (1/\lambda > 0)$$

converge uniformly in the interval $A \leq a \leq B$;

$$\text{(iii)} \quad (A)\text{-}\int_c^\infty f(x, a)dx$$

converge uniformly in $A \leq a \leq B$, *then the last integral defines a function*

$$\varphi(a) = (A)\text{-}\int_0^\infty f(x, a)dx$$

continuous in that interval.

Proof. We take any point a_0 in the interval $A \leq a \leq B$. As the integral is uniformly convergent,

$$\varphi(a_0) = \frac{1}{\lambda'}\int_0^\infty dx.e^{-x/\lambda'}\int_c^x f(t, a_0)dt + R_{\lambda'}(a_0)$$

and

$$\varphi(a_0+h) = \frac{1}{\lambda'}\int_0^\infty dx.e^{-x/\lambda'}\int_c^x f(t, a_0+h)dt + R_{\lambda'}(a_0+h)$$

where both $|R_{\lambda'}(a_0)| < \epsilon$ and $|R_{\lambda'}(a_0+h)| < \epsilon$ when $\lambda' > \Lambda(\epsilon)$. Hence

$$|\varphi(a_0+h) - \varphi(a_0)| \leq \left|\frac{1}{\lambda'}\int_0^\infty dx.e^{-x/\lambda'}\int_c^x [f(t, a_0+h) - f(t, a_0)]dt\right| + |R_{\lambda'}(a_0+h)| + |R_{\lambda'}(a_0)|$$

We now keep λ' fixed. Then, since, by virtue of (ii)

$$\frac{1}{\lambda'}\int_0^\infty dx.e^{-x/\lambda'}\int_c^x f(t, a)dt$$

is a continuous function of a in the interval $A \leq a \leq B$, it follows that

$$\left|\frac{1}{\lambda'}\int_0^\infty dx.e^{-x/\lambda'}\int_c^x [f(t, a_0+h) - f(t, a_0)]dt\right| < \epsilon, \quad |h| < \delta.$$

Hence

$$|\varphi(a_0+h) - \varphi(a_0)| < 3\epsilon, \quad |h| < \delta$$

which proves the proposition.

INTEGRATION WITH RESPECT TO A PARAMETER: REVERSAL OF THE ORDER OF INTEGRATION

Theorem 2. *Let*

(i) $f(x, a)$ *be continuous in the region* $c \leq x \leq \infty$, $A \leq a \leq B$;

$$\text{(ii)} \quad \frac{1}{\lambda}\int_0^\infty e^{-x/\lambda}\int_c^x f(t, a)dt, \quad (1/\lambda > 0)$$

converge uniformly in the interval $A \leq a \leq B$;

$$\text{(iii)} \quad (A)\text{-}\int_0^\infty f(x, a)dx$$

converge uniformly in the interval $A \leq a \leq B$;

$$\varphi(a) = (A)\text{-}\int_0^\infty f(x, a)\,dx,$$

then

$$\int_{a_0}^{a_1} \varphi(a)\,da = (A)\text{-}\int_c^\infty dx \int_{a_0}^{a_1} f(x, a)\,da$$

or

$$\int_{a_0}^{a_1} da.(A)\text{-}\int_c^\infty f(x, a)\,dx = (A)\text{-}\int_c^\infty dx \int_{a_0}^{a_1} f(x, a)\,da.$$

Proof. The function $\varphi(a)$ is continuous by Theorem 1 and therefore integrable. Because of uniform convergence

$$\varphi(a) = \frac{1}{\lambda'}\int_0^\infty dx e^{-x/\lambda'}\int_c^x f(t, a)\,dt + R_{\lambda'}(a),$$

where $|R_{\lambda'}(a)| < \epsilon$, when $\lambda' \geq \Lambda(\epsilon)$.

$$\int_{a_0}^{a_1} \varphi(a)\,da = \frac{1}{\lambda'}\int_{a_0}^{a_1} da \int_0^\infty dx.e^{-x/\lambda'}\int_c^x f(t, a)\,dt + \int_{a_0}^{a_1} R_{\lambda'}(a)\,da$$

We keep λ' fixed. The order of integration in the first integral on the right can be reversed, by virtue of (ii), so that

$$\int_{a_0}^{a_1} da.(A)\text{-}\int_c^\infty f(x, a)\,dx - \frac{1}{\lambda'}\int_0^\infty dx.e^{-x/\lambda'}\int_c^x dt \int_{a_0}^{a_1} f(t, a)\,da = \int_{a_0}^{a_1} R_{\lambda'}(a)\,da.$$

Hence

$$\left|\int_{a_0}^{a_1} da.(A)\text{-}\int_c^\infty f(x, a)\,dx - \frac{1}{\lambda'}\int_0^\infty dx.e^{-x/\lambda'}\int_c^x dt \int_{a_0}^{a_1} f(t, a)\,da\right| < \epsilon\,|a_1 - a_0|$$

for $\lambda' \geq \Lambda(\epsilon)$. Therefore

$$\int_{a_0}^{a_1} da.(A)\text{-}\int_c^\infty f(x, a)\,dx = (A)\text{-}\int_c^\infty dx \int_{a_0}^{a_1} f(x, a)\,da.$$

DIFFERENTIATION WITH RESPECT TO A PARAMETER

Theorem 3. *Let*

(i) $(A)\text{-}\int_c^\infty f(x, a)\,dx$

converge in the interval $A \leq a \leq B$:

$$\varphi(a) = (A)\text{-}\int_c^\infty f(x, a)\,dx;$$

(ii) $\partial f/\partial a$ *exist*;

(iii) *the function* $\partial f/\partial a = f_a(x, a)$

be continuous in the region $c \leq x \leq \infty$, $A \leq a \leq B$;

$$\text{(iv)} \quad \frac{1}{\lambda}\int_0^\infty e^{-x/\lambda}\int_c^x f_a(t, a)dt, \quad (1/\lambda > 0)$$

converge uniformly in the interval $A \leq a \leq B$;

$$\text{(v)} \quad (A)\text{-}\int_c^\infty f_a(x, a)dx$$

converge uniformly in the interval $A \leq a \leq B$, *then the function* $\varphi(a)$ *has a derivative, the derivative is continuous and it is given by the last integral*:

$$\frac{d\varphi}{da} = (A)\text{-}\int_c^\infty f_a(x, a)dx$$

Proof. Let

$$\psi(a) = (A)\text{-}\int_c^\infty f_a(x, a)dx.$$

Then $\psi(a)$ is continuous and therefore integrable. By Theorem 2,

$$\int_A^a \psi(a)da = (A)\text{-}\int_c^\infty dx\int_A^a \frac{\partial f}{\partial a}da = (A)\text{-}\int_c^\infty \{f(x, a) - f(x, A)\}dx$$

$$= (A)\text{-}\int_c^\infty f(x, a)dx - (A)\text{-}\int_c^\infty f(x, A)dx.$$

Hence

$$\int_A^a \psi(a)da = \varphi(a) - \varphi(A).$$

From the continuity of $\psi(a)$, it at once follows that

$$\psi(a) = d\varphi/da.$$

The three theorems established above can immediately be generalised for any number of parameters.

Let us now generalise the concept of the volume integral:

$$\int f(x_1, x_2, x_3)dx_1dx_2dx_3$$

taken over all space. We first integrate over a sphere of radius r, the centre being at a fixed point, so that the integral is now a function of r only *viz*.,

$$\int f(x_1, x_2, x_3)dx_1dx_2dx_3 = F(r)$$

If now the (A)-limit of $F(r)$ as $r \to \infty$ exists we say that the integral exists and its value is

$$(A)\text{-}\int f(x_1, x_2, x_3)dx_1dx_2dx_3 = (A)\text{-}\lim_{r\to\infty} F(r). \tag{4}$$

We may similarly generalise the concept when the integrand contains parameters:

$$\int f(a_1, a_2, \ldots, a_r, x_1, x_2, x_3)dx_1dx_2dx_3$$

taken over all space, as has been done in a previous paper (Ghosh, 1947). In every case there remains an arbitrariness due to the choice of the centre of sphere of integration. This arbitrariness can be removed by other conditions of the problem.

We may proceed to study the equation

$$\Delta\varphi_q(\mathbf{r}) = -4\pi \exp[i(\mathbf{qr})] \tag{5}$$

as in our previous paper (Ghosh, 1947, p. 25). Reasoning on the same lines and taking (A)-limit instead of $(C, 1)$-limit, we arrive at the solution

$$\varphi_q(\mathbf{r}) = (A)\text{-}\int \frac{\exp[i(\mathbf{qR})]}{|\mathbf{R}-\mathbf{r}|} d\Omega_R = \frac{4\pi}{q^2}\exp[i(\mathbf{qr})]. \tag{6}$$

APPLICATIONS

For the purpose of applications we have to generalise the usual rules of Quantum Mechanics for the calculation of the matrix-elements of an operator A, *viz.*,

$$A_{mn} = (A)\text{-}\int \psi_m^* A\psi_n d\tau$$

instead of

$$A_{mn} = \int \psi_m^* A\psi_n d\tau$$

taken over the entire configuration space where ψ_m and ψ_n are eigenfunctions belonging to the m-th and n-th state respectively.

In the solution of problems of Quantum Mechanics such integrals occur which in the last analysis reduce to

$$\int^\infty k_2^{2n}\cos k_0 a dk_0, \quad \int^\infty k_0^{2n+1}\sin k_0 a dk_0,$$

$$\int^\infty k_0^{2n+1}\cos k_0 a dk_0, \quad \int^\infty k_0^{2n}\sin k_0 a dk_0$$

[*Cf.* Dirac (1948)]. The Abel limit of these integrals are

$$(A)\text{-}\int_0^\infty k_0^{2n}\cos k_0 a dk_0 = 0, \quad (A)\text{-}\int_0^\infty k_0^{2n+1}\sin k_0 a dk_0 = 0$$

$$(A)\text{-}\int_0^\infty k_0^{2n+1}\cos k_0 a dk_0 = (-1)^{n+1}(2n+1)!/a^{2n+2},$$

$$(A)\text{-}\int_0^\infty k_0^{2n}\sin k_0 a\,dk_0 = (-1)^n.(2n)!/a^{2n+1}.$$

These calculations may be made directly from definition but the results are quickly obtained from

$$(A)\text{-}\int_0^\infty \cos k_0 a\,dk_0 = 0, \quad (A)\text{-}\int_0^\infty \sin k_0 a\,dk_0 = 1/a$$

by differentiating with respect to the parameter a, the operation being valid in the light of Theorem 3. The concept of Abel-limit ascribes to the above integrals the same values as those ascribed to them by the concept of Cesàro limit (Ghosh, 1948). This agreement is not accidental but is a consequence of an established proposition (*Cf.* Hardy, 1949, Art 6.8).

Scattering by a centre of force. Born's Approximation.

Our problem is to calculate the scattering of a beam of particles $[\exp(ikz)]$ by a field $V(r)$. We shall obtain an approximate formula.

We have to solve the wave equation

$$\Delta\psi + [k^2 - U(r)]\psi = 0, \tag{7}$$

where $k^2 = 8\pi^2 mE/h^2$, $U(r) = 8\pi^2 mV(r)/h^2$ and where ψ must have the asymptotic form

$$\psi \sim e^{ikz} + r^{-1}e^{ikr}f(\theta). \tag{8}$$

We know that the most general bounded solution of the equation

$$\Delta\psi + k^2\psi = \Phi(\mathbf{r})$$

is

$$\psi = G(\mathbf{r}) - \frac{1}{4\pi}\int \frac{\exp[ik|\mathbf{r}-\mathbf{r}'|]}{|\mathbf{r}-\mathbf{r}'|}\Phi(\mathbf{r}')d\tau' \tag{9}$$

where G is the general solution of

$$\Delta G + k^2 G = 0.$$

We can at once generalise (9) in the sense of Abel convergence, *viz.*,

$$\psi = G(\mathbf{r}) - \frac{1}{4\pi}(A)\text{-}\int \frac{\exp[ik|\mathbf{r}-\mathbf{r}'|]}{|\mathbf{r}-\mathbf{r}'|}\Phi(\mathbf{r}')d\tau'. \tag{9'}$$

Here we take the centre of the sphere of integration at the origin, and this leads to the prescribed asymptotic form.

The general solution of (7) satisfies the integral equation

$$\psi = G - \frac{1}{4\pi}(A)\text{-}\int \frac{\exp[ik|\mathbf{r}-\mathbf{r}'|]}{|\mathbf{r}-\mathbf{r}'|}U(r')\psi(\mathbf{r}')d\tau'. \tag{10}$$

The expression on the right of (10) has to represent an outgoing wave. In order that ψ may have the prescribed asymptotic form (8), we must choose

$$G = e^{ikz}.$$

To obtain $f(\theta)$, we require the asymptotic form of (9′) for large r. If $\mathbf{n}$ be a unit vector in the direction of $\mathbf{r}$, we have

$$|\mathbf{r}-\mathbf{r}'| \sim \mathbf{r}-(\mathbf{n}\,\mathbf{r}') + \text{terms of the order } 1/r.$$

Hence from (9′)

$$\psi \sim e^{ikz} - r^{-1}e^{ikr}\frac{1}{4\pi}(A)\text{-}\int \exp\left[-ik(\mathbf{n}\,\mathbf{r}')\right] \mathsf{U}(r')\psi(\mathbf{r})d\tau' \tag{11}$$

To obtain a formula for $f(\theta)$, we assume that the wave is not much diffracted by the scattering centre, so that we can replace $\psi(\mathbf{r}')$ in the integral (11) by the unperturbed wave function e^{ikz}.

We then obtain from (8) and (11)

$$f(\theta) = -\frac{1}{4\pi}(A)\text{-}\int \exp\left[ik(\mathbf{n}_0-\mathbf{n},\ \mathbf{r}')\right] \mathsf{U}(r')d\tau' \tag{12}$$

where $\mathbf{n}_0$ is the unit vector along the z-axis, so that $z = (\mathbf{n}_0\mathbf{r})$.

For the Coulomb field

$$\mathsf{U}(r) = \frac{8\pi^2 me}{h^2 r}$$

and hence

$$f(\theta) = -\frac{2\pi me}{h^2}(A)\text{-}\int \frac{\exp\left[ik(\mathbf{n}_0-\mathbf{n},\ \mathbf{r}')\right]}{r'}d\tau' = -\frac{8\pi^2 me}{h^2}\frac{1}{k^2|\mathbf{n}_0-\mathbf{n}|^2} \tag{13}$$

by (6).

Now $|\mathbf{n}_0-\mathbf{n}| = 2\sin\frac{1}{2}\theta$. Therefore

$$f(\theta) = -\frac{4\pi^2 me}{h^2k^2\sin^2\frac{1}{2}\theta}.$$

The intensity scattered into the solid angle $d\omega$ is $|f(\theta)|^2 d\omega$, and this gives the Rutherford Scattering Formula.

Here with the help of Abel-convergence of infinite integrals we have arrived at the result smoothly. If some such idea is not brought in, the right hand side of (13) oscillates as the limit of integration tends to infinity. To overcome this difficulty Wentzel (1927) had to use a modified field

$$V = \frac{e}{r}\exp(-r/R)$$

instead of a Coulomb field and then neglect the modification under certain physical assumptions. From the mathematical stand point this is equivalent to the introduction of a 'convergence factor'

$$e^{-pr},\quad p>0$$

in the expression

$$\int_0^\infty \sin qr dr$$

and after integration let $p \to 0$. Thus

$$\int_0^\infty dr.e^{-pr} \sin qr = q/(p^2+q^2) \to 1/q.$$

as $p \to 0$, and to take this as the value of the integral (Sommerfeld, 1939). This is artificial in as much as the 'convergence factor' is brought in only in the last step to save the situation without enquiring into the effect of its appearance and ultimate disappearance on the preceding steps. Here we have developed a theory in which the notion of the 'convergence factor' is introduced from the very outset and hardly any special hypothesis becomes necessary afterwards.

Another example of the application of our theory is furnished by Bethe's investigation regarding the passage of fast particles through matter (Bethe, 1930). There proceeding by the method of variation of parameters, Bethe had to calculate matrix-elements which involved integrals of the type

$$\int \frac{\exp\,[i(\mathbf{qr})]}{|\mathbf{R}-\mathbf{r}|}\, d\Omega_R$$

taken over all space. This integral does not exist in the ordinary sense. Bethe evaluates it by an unjustifiable interchange of limit processes. In the light of our discussion on (A)-convergent integrals, together with the boundary condition and generalised rule for calculating the matrix-elements we are at once led to the calculation of integrals of the type

$$(A)\text{-}\int \frac{\exp\,(i\mathbf{qr})}{|\mathbf{R}-\mathbf{r}|}\, d\Omega_R = \frac{4\pi}{q^2} \exp\,[i(\mathbf{qr})]$$

by (6) subject to certain conditions (*Cf.* Ghosh, 1947).

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ON THE FLEXURE PROBLEM OF A LIMACON AND SOME OTHER BOUNDARIES

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1. The object of the present paper is to derive a solution of the flexure problem of a beam analogous to the formal solution of Morris (1940, p. 81), when the function $z = \omega(\zeta)$ which maps the cross-section of the beam on the unit circle in the ζ-plane is a polynomial or a power series in ζ. Morris's procedure is defective unless the mapping function is a polynomial, in which case it leads to the correct function-theoretic solution of the problem. In Morris's paper, $\zeta = \xi + i\eta$ is introduced in a different sense, *viz.*, that the curve $\eta = 0$ in the net defined by the transformation $z = \sum_{n=0}^{\infty} a_n \exp(ni\zeta)$ gives the boundary of the cross-section. The formal solution is then constructed with the help of two functions $z(\zeta)$ and $\bar{z}(\zeta)$. But it is seen at once that the series for $z(\zeta)$ and $\bar{z}(\zeta)$ cannot both converge in the same domain. The method of solution followed in this paper is free from this objection and leads automatically to a simple sufficient condition (*viz.*, that the radius of convergence of the power series for $\omega(\zeta)$ is > 1) for the formal solution to be a function-theoretic one. This simple condition appears to have been overlooked by Sokolnikoff (1946, p. 173) in his derivation of Morris's formal solution of the torsion problem from Muschelisvili's formula. Higgins (1942) has applied Morris's formula to obtain the correct solution of the torsion problem for the inverse of an ellipse with respect to its centre and his success is due to the fact that the radius of convergence of the power series for $\omega(\zeta)$ is in this case > 1.

As typical applications of the formulæ derived in this paper, two boundaries are considered: (1) an elliptic limacon and (2) inverse of an ellipse with respect to its centre. The first case has been considered by Stevenson (1939, p. 126), but his result (6.2) for the centre of flexure does not agree with that obtained in this paper. A scrutiny of his paper shews that an error (an unnecessary factor 2) has crept into the first line of the formula (3.26) of his paper. This error is responsible for the inaccuracy in his final result. The second case has been considered by the present author by a different method.

2. The canonical flexure functions f_{11}, f_{21}, f_{12} and f_{22} are given by (Ghosh, 1947, p. 10),

$$f_{11}+if_{21} = -\frac{i}{4\pi}\int_{\gamma} [\omega(\sigma)]^2\, \bar{\omega}(1/\sigma)\frac{d\sigma}{\sigma-\zeta}, \qquad (2.1)$$

$$f_{11}-if_{21} = \frac{i}{4\pi}\int_{\gamma} \omega(\sigma)[\bar{\omega}(1/\sigma)]^2\,\frac{d\sigma}{\sigma-\zeta}, \qquad (2.2)$$

$$f_{12}+if_{22} = -\frac{i}{2\pi}\int^{\zeta} d\zeta \int_{\gamma} [\omega(\sigma)]^2 \bar{\omega}'(1/\sigma)\frac{d\sigma}{\sigma^2(\sigma-\zeta)}, \tag{2.3}$$

$$f_{12}-if_{22} = -\frac{i}{2\pi}\int^{\zeta} d\zeta \int_{\gamma} [\bar{\omega}(1/\sigma)]^2 \omega'(\sigma)\frac{d\sigma}{\sigma-\zeta}, \tag{2.4}$$

where $z = \omega(\zeta)$ maps conformally the area of the cross-section on the unit circle bounded by γ in the ζ-plane. Let us suppose that $\omega(\zeta)$ can be expanded in a power series of the form,

$$\omega(\zeta) = \sum_0^\infty a_n\zeta^n. \tag{2.5}$$

If the radius of convergence R of the power series be >1, then the series (2.5) and the series

$$\bar{\omega}(1/\zeta) = \sum_0^\infty \bar{a}_n/\zeta^n \tag{2.6}$$

are absolutely and uniformly convergent in the domain $1/R < |\zeta| < R$ of the ζ-plane. Therefore, multiplication of the power series, re-arrangement of terms of the product series and integration of the product series along γ are permissible operations. We have,

$$[\omega(\sigma)]^2 = \sum_0^\infty b_n\sigma^n \tag{2.7}$$

where

$$b_n = (a_0a_n + a_1a_{n-1} + \cdots + a_na_0), \tag{2.8}$$

Also,

$$[\bar{\omega}(1/\sigma)]^2 = \sum_0^\infty \bar{b}_n/\sigma^n \tag{2.9}$$

and hence,

$$[\omega(\sigma)]^2\omega(1/\sigma) = \sum_0^\infty c_n\sigma^n + \sum_1^\infty d_n/\sigma^n$$

where

$$c_n = \sum_{r=0}^\infty b_{n+r}\bar{a}_r, \tag{2.10}$$

$$d_n = \sum_{r=0}^\infty b_r\bar{a}_{n+r}. \tag{2.11}$$

Substituting in (2.1) and using Cauchy's Residue Theorem we get at once,

$$f_{11}+if_{21} = \tfrac{1}{2}\sum_0^\infty c_n\zeta^n. \tag{2.12}$$

Also we have,

$$\omega(\sigma)[\bar{\omega}(1/\sigma)]^2 = \sum_0^\infty \bar{d}_n\sigma^n + \sum_1^\infty \bar{c}_n/\sigma^n$$

then (2.2) gives,

$$f_{11}-if_{21} = -\tfrac{1}{2}\sum_0^\infty \bar{d}_n\zeta^n. \tag{2.13}$$

Finally,

$$[\omega(\sigma]^2\bar{\omega}'(1/\sigma) = \sum_0^\infty p_n\sigma^n + \sum_1^\infty q_n/\sigma^n,$$

$$[\bar{\omega}(1/\sigma)]^2\omega'(\sigma) = \sum_0^\infty \bar{q}_{n+2}\sigma^n + \sum_1^\infty \bar{p}_{n-2}/\sigma^n$$

where,

$$p_n = \sum_{r=1}^{\infty} r b_{n+r+1} \bar{a}_r, \tag{2.14}$$

$$q_n = \sum_{r=0}^{\infty} (n+r-1) b_r \bar{a}_{n+r-1}. \tag{2.15}$$

Substituting in (3.3) and (3.4) we get,

$$f_{12} + i f_{22} = \sum_{1}^{\infty} p_{n-1} \zeta^n / n, \tag{2.16}$$

$$f_{12} - i f_{22} = \sum_{1}^{\infty} \bar{q}_{n+1} \zeta^n / n. \tag{2.17}$$

These formulæ are analogous to the formulæ given in Morris's formal solution for the flexure problem.

3. The flexural moments N_1 and N_2 are given by (Ghosh 1947, p. 12)

$$N_1 = J_{11} + \tfrac{1}{4}(1-2\sigma) J_{21}, \quad N_2 = J_{21} - \tfrac{1}{4}(1-2\sigma) J_{22} \tag{3.1}$$

where,

$$J_{11} = \tfrac{1}{2}\mathcal{R}\int_{\gamma} \omega(\zeta)\bar{\omega}(1/\zeta) df_{11}(\zeta) + \tfrac{1}{2}(1+\sigma)\mathcal{R}\int_{\gamma} \omega(\zeta)\bar{\omega}(1/\zeta) df_{22}(\zeta), \tag{3.2}$$

$$J_{21} = \tfrac{1}{2}\mathcal{R}\int_{\gamma} \omega(\zeta)\bar{\omega}(1/\zeta) df_{21}(\zeta) + \tfrac{1}{2}(1+\sigma)\mathcal{R}\int_{\gamma} \omega(\zeta)\bar{\omega}(1/\zeta) df_{22}(\zeta). \tag{3.3}$$

$$J_{12} - i J_{22} = -\tfrac{1}{4}\int_{\gamma} [\omega(\zeta)\bar{\omega}(1/\zeta)]^2 d\omega(\zeta) \tag{3.4}$$

and the co-ordinates of the centre of flexure are given by (Ghosh, 1947, p. 7)

$$x_f = \frac{I_{11}N_2 - I_{12}N_1}{2(1+\sigma)(I_{11}I_{22} - I_{12}^2)}, \qquad y_f = \frac{I_{12}N_2 - I_{22}N_1}{2(1+\sigma)(I_{11}I_{22} - I_{12}^2)}. \tag{3.5}$$

From (2.12), (2.13), (2.5) and (2.6)

$$\int_{\gamma} \omega(\zeta)\bar{\omega}(1/\zeta) df_{11}(\zeta) = \tfrac{1}{2}\pi i \sum_{1}^{\infty} \bar{r} k_r (c_r - \bar{d}_r), \tag{3.6}$$

$$\int_{\gamma} \omega(\zeta)\bar{\omega}(1/\zeta) df_{21}(\zeta) = \tfrac{1}{2}\pi \sum_{1}^{\infty} r k_r (c_r + d_r), \tag{3.7}$$

where

$$k_n = \sum_{r=0}^{\infty} \bar{a}_{n+r} a_r. \tag{3.8}$$

Also, from (2.16), (2.17), (2.5) and (2.6),

$$\int_{\gamma} \omega(\zeta)\bar{\omega}(1/\zeta) df_{12}(\zeta) = \pi i \sum_{1}^{\infty} k_r (p_{r-1} + \bar{q}_{r+1}), \tag{3.9}$$

$$\int_{\gamma} \omega(\zeta)\bar{\omega}(1/\zeta) df_{22}(\zeta) = \pi \sum_{1}^{\infty} k_r (p_{r-1} - \bar{q}_{r+1}) \tag{3.10}$$

From (3.5), (3.7), (3.9) and (3.4), we have

$$J_{12}-iJ_{22} = -\tfrac{1}{2}\pi i\sum_{1}^{\infty} rl_r a_r \tag{3.11}$$

where

$$l_n = \sum_{r=0}^{\infty} \bar{b}_{n+r} b_r. \tag{3.12}$$

Therefore

$$J_{11} = \tfrac{1}{4}\pi\mathcal{R}\left\{i\sum_{1}^{\infty} rk_r(c_r-\bar{d}_r)+2i(1+\sigma)\sum_{1}^{\infty} k_r(p_{r-1}+\bar{q}_{r+1})\right\}, \tag{3.13}$$

$$J_{21} = \tfrac{1}{4}\pi\mathcal{R}\left\{\sum_{1}^{\infty} rk_r(c_r+\bar{d}_r)+2(1+\sigma)\sum_{1}^{\infty} k_r(p_{r-1}-\bar{q}_{r+1})\right\}. \tag{3.14}$$

4. As an example, we consider the limacon $r = b-2a\cos\theta$ which is mapped conformally on the unit circle by the formula

$$z = -a+b\zeta-a\zeta^2 \tag{4.1}$$

z_0, the coordinate of the centroid of the cross-section is,

$$z_0 = -2a(a^2+b^2)/(b^2+2a^2). \tag{4.2}$$

Transferring the origin to the centroid, the transformation formula becomes

$$z = b\zeta-a\zeta^2+\frac{ab^2}{b^2+2a^2}. \tag{4.3}$$

In this case,

$$a_0 = ab^2/(b^2+2a^2), \quad a_1 = b, \quad a_2 = -a, \tag{4.4}$$

$$b_0 = q_0^2, \quad b_1 = 2a_0 b, \quad b_2 = b^2-2aa_0, \quad b_3 = -2ab, \quad b_4 = a^2, \tag{4.5}$$

$$c_0 = a_0(a_0^2+b^2), \quad c_1 = (2a_0^2 b-2a_0 ab+2a^2 b+b^3),$$

$$c_2 = -2aa_0^2+b^2a_0-2ab^2-a^3, \quad c_3 = -2aba_0+a^2b, \quad c_1 = a^2a_0, \tag{4.6}$$

$$d_0 = a_0^3+a_0b^2, \quad d_1 = ba_0^2-2aa_0b, \quad d_2 = -aa_0^2, \tag{4.7}$$

$$p_0 = b(b^2-2aa_0+4a^2), \quad p_1 = -2a(a^2+b^2), \quad p_2 = a^2b, \tag{4.8}$$

$$q_2 = ba_0^2-4aba_0, \quad q_3 = -2aa_0^2, \tag{4.9}$$

$$k_1 = b(a_0-a), \quad k_2 = -aa_0, \tag{4.10}$$

$$l_1 = 2a_0^3b-4a_0^2ab-2a^3b, \quad l_2 = -2aa_0^3+b^2a_0^2-8aa_0b^2, \tag{4.11}$$

all other a's, b's, c's, d's, p's, q's, k's, l's being zero.

Substituting the above values, the flexural functions are obtained. The flexural moments are found to be

$$N_1 = 0, \quad N_2 = -\pi a_0 a^4(3+4\sigma). \tag{4.12}$$

The moments and products of inertia of the cross-section with respect to the coordinate axes are

$$I_{11} = \tfrac{1}{4}\pi(-4ab^2a_0+b^4+6a^2b^2+2a^4), \quad I_{22} = \tfrac{1}{4}\pi(b^4+6a^2b^2+2a^4), \quad I_{12} = 0. \tag{4.13}$$

Hence the coordinates of the centre of flexure are

$$x_f = \frac{-a^5b^2(3+4\sigma)}{(1+\sigma)(b^2+2a^2)(b^4+6a^2b^2+2a^4)}, \quad y_f = 0. \tag{4.14}$$

This result does not agree with that of Stevenson (1939) whose results are found to be incorrect.

If we put $b = 2a$, we then get the case of a cardioid and the results agree with known ones.

5. As a second example, we consider the cross-section in the form of an inverse of an ellipse with respect to its centre. The mapping function is given by (Sokolnikoff, p. 184)

$$z = \omega(\zeta) = \frac{k\zeta}{a^2+\zeta^2} = \sum_0^\infty a_n\zeta^n, \quad (a > 1) \tag{5.1}$$

where

$$a_n = 0 \text{ for } n = 0, 2, 4, 6, \ldots \text{ and } a_n = (-1)^{\frac{1}{2}(n-1)}k/a^{n+1} \text{ for } n = 1, 3, 5, \ldots \tag{5.2}$$

and the centroid is taken as the origin. The radius of convergence of the series (5.1) is obviously > 1.

From (2.8),

$$b_n = (-1)^{\frac{1}{2}(n-2)}n.k^2/(2a^{n+2}) \text{ for } n = 0, 2, 4, 6, \ldots, \quad b_n = 0 \text{ for } n = 1, 3, 5, \ldots \tag{5.3}$$

From (2.10),

$$c_n = (-1)^{\frac{1}{2}(n-1)}\frac{k^3}{2a^2}\sum_{r=1}^\infty \frac{n+r}{a^{n+r}}.\frac{1}{a^{r+1}} \text{ for } n = 1, 3, 5, \ldots, \quad c_n = 0 \text{ for } n = 0, 2, 4, 6, \ldots \tag{5.4}$$

From (2.11),

$$d_n = (-1)^{\frac{1}{2}(n+1)}\frac{k^3}{a^{n-1}(a^4-1)^3} \text{ for } n = 1, 3, 5, \ldots, \qquad d_n = 0 \text{ for } n = 0, 2, 4, 6, \ldots \tag{5.5}$$

From (2.14),

$$p_n = (-1)^{\frac{1}{2}n}\frac{k^3}{2a^{n+2}}\sum_{r=1}^\infty (2r-1)(n+2r)/a^{4r}$$

$$\text{for } n = 0, 2, 4, 6, \ldots, \quad p_n = 0 \text{ for } n = 1, 3, 5, \ldots \tag{5.6}$$

From (2.15),

$$q_n = (-1)^{\frac{1}{2}n}k^3\left[\frac{n-2}{a^{n-2}(a^4-1)^2}+\frac{1}{a^{n+2}}\sum_{r=1}^\infty \frac{r(2r+1)}{a^{4r}}\right]$$

$$\text{for } n = 0, 2, 4, 6, \ldots, \quad q_n = 0 \text{ for } n = 1, 3, 5, \ldots \tag{5.7}$$

Substituting the above values in (2.12), (2.13), (2.16), (2.17) we get the values of canonical flexure functions in a series form. Their values in closed form can be obtained by summing the series by suitable rearrangement of its terms and will be found to agree with the results obtained in a previous paper (Mitra, 1948).

In conclusion, I express my gratefulness to Dr. S. Ghosh for his helpful suggestions in the preparation of this paper.

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A NOTE ON THE EXPRESSION FOR THE SAMPLE ESTIMATE OF THE COEFFICIENT OF PARTIAL CORRELATION

By

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This note claims nothing new but merely presents old results in a form which would be more suitable for the direct calculation of the coefficient of partial correlation drawn from a multivariate population.

Let the sample of size n drawn from a population with k variates $x_1, x_2, \ldots, x_k$ be denoted by

$$(x_{1l}, x_{2l}, \ldots, x_{kl}), \quad (l = 1, 2, \ldots, n) \tag{1}$$

Then the sample estimate of the total correlation coefficient r_{ij} between x_i and x_j is

$$r_{ij} = \frac{p_{ij}}{\sqrt{(p_{ii}p_{jj})}}, \tag{2}$$

where p_{ij} is the product moment

$$\left.\begin{aligned} p_{ij} &= \sum_{l=1}^{n} x_{il}x_{jl} - \frac{1}{n}\sum_{l=1}^{n} x_{il}\sum_{l=1}^{n} x_{jl} \\ &= \sum_{l=1}^{n} X_{il}X_{jl}, \end{aligned}\right\} \tag{3}$$

X_{il} being the deviation of x_{il} from the mean of x_i, i.e,

$$X_{il} = x_{il} - x_i.$$

If P_{ij} is the minor of p_{ij} in the determinant

$$\Delta = \|p_{ij}\|,$$

then the sample estimate of the multiple correlation coefficient between x_i and the other variates may be expressed in the form (Kendall, 1948)

$$R_{i.12\ldots(i-1)(i+1)\ldots k} = \sqrt{\left(1 - \frac{\Delta}{p_{ii}P_{ii}}\right)}. \tag{4}$$

The partial correlation coefficient also can be elegantly expressed in terms of the product moments. This expression is going to be established in the present note.

The partial correlation coefficient between x_1 and x_2 when the other variates are kept constant may be defined as

$$r_{12.34\ldots k} = \sqrt{(b_{12.34\ldots k} \times b_{21.34\ldots k})}, \tag{5}$$

where $b_{12.34\ldots k}$ is the coefficient of partial regression of x_1 on x_2, i.e., the coefficient of x_2

in the multiple regression equation

$$x_1 = b_{10} + b_{12.34\ldots k}\,x_2 + b_{13.24\ldots k}\,x_3 + \cdots + b_{1k.23\ldots(k-1)}\,x_k, \tag{6}$$

and similarly $b_{21.34\ldots k}$ is the coefficient of partial regression of x_2 on x_1.

Measuring the variates about their respective means, the regression equation (6) becomes

$$X_1 = b_{12.34\ldots k}\,X_2 + b_{13.24\ldots k}\,X_3 + \cdots + b_{1k.23\ldots(k-1)}\,X_k. \tag{6'}$$

The values of the coefficients in the above equation are to be estimated from the same sample by the principle of least squares, *i.e.*, by minimising

$$L = \sum_{i=1}^{n}(X_{1i} - b_{12.34\ldots k}\,X_{2i} - b_{13.24\ldots k}\,X_{3i} - \cdots - b_{1k.23\ldots(k-1)}\,X_{ki})^2.$$

The conditions that the above expression would be a minimum are

$$\frac{\partial L}{\partial b_{12.34\ldots k}} = \frac{\partial L}{\partial b_{13.24\ldots k}} = \cdots = \frac{\partial L}{\partial b_{1k.23\ldots(k-1)}} = 0.$$

These lead to the $(k-1)$ normal equations

$$\left.\begin{array}{l}
b_{12.34\ldots k}\,p_{22} + b_{13.24\ldots k}\,p_{23} + \cdots + b_{1k.23\ldots(k-1)}\,p_{2k} = p_{21}, \\
b_{12.34\ldots k}\,p_{32} + b_{13.24\ldots k}\,p_{33} + \cdots + b_{1k.23\ldots(k-1)}\,p_{3k} = p_{31}, \\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
b_{12.34\ldots k}\,p_{k2} + b_{13.24\ldots k}\,p_{k3} + \cdots + b_{1k.23\ldots(k-1)}\,p_{kk} = p_{k1}.
\end{array}\right\} \tag{7}$$

Solution of the above linear equations for the b's by the method of determinants gives

$$b_{12.34\ldots k} = \frac{\begin{vmatrix} p_{21} & p_{23} & \cdots & p_{2k} \\ p_{31} & p_{33} & \cdots & p_{3k} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ p_{k1} & p_{k3} & \cdots & p_{kk} \end{vmatrix}}{\begin{vmatrix} p_{22} & p_{23} & \cdots & p_{2k} \\ p_{32} & p_{33} & \cdots & p_{3k} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ p_{k2} & p_{k3} & \cdots & p_{kk} \end{vmatrix}} = \frac{P_{12}}{P_{11}}. \tag{8}$$

Similarly by using the principle of least squares in the regression equation of x_2 on the others, we get

$$b_{21.34\ldots k} = \frac{P_{21}}{P_{22}}. \tag{9}$$

Hence, substituting (8) and (9) in (5), we get

$$r_{12.34\ldots k} = \frac{P_{12}}{\sqrt{(P_{11}P_{22})}}. \tag{10}$$

This is the expression that we seek to establish. The expression is analogous to the expression (3) for the total correlation coefficient,—only that the p's in (3) are replaced by their minors P's. The utility of the expression (10) lies in this fact, as well as in that this gives the direct expression for the coefficient of partial correlation between two variates in a population with several variates, and the numerical calculation of the coefficient for a sample is somewhat simplified than by using the formula

$$r_{12.34\ldots j} = \frac{r_{12.4\ldots j} - r_{13.4\ldots j}\, r_{23.4\ldots j}}{\sqrt{\{(1-r^2_{13.4\ldots j})(1-r^2_{23.4\ldots j})\}}} \tag{11}$$

by stages.

Of course the expressions (4) and (10) for the coefficients of multiple and partial correlation might also be given in terms of the correlation matrix

$$\lambda = \|r_{ij}\| \tag{12}$$

instead of the product moment matrix $\Delta = \|p_{ij}\|$. The expressions would be

$$R_{1.23\ldots k} = \sqrt{\left(1 - \frac{\lambda}{\lambda_{11}}\right)}, \tag{4'}$$

and

$$r_{12.34\ldots k} = \frac{\lambda_{12}}{\sqrt{(\lambda_{11}\,\lambda_{22})}}, \tag{10'}$$

where λ_{ij} is the minor of r_{ij} in λ.

The expressions (4) and (4′) are easily deducible from one another, as also are the expressions (10) and (10′) by the substitution of (2) and by remembering that $r_{ii} = 1$.

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ON THE BENDING OF AN ELASTIC PLATE—I*

By

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Happel (1921) determined the deflexion of the central plane of a thin elliptic plate, made of isotropic elastic material, loaded at the centre, and clamped or supported at the edge. S. Ghosh (1924) determined the deflexion when the load is placed at a focus, the edge being clamped. Later on the author (1948) determined the deflexion when the load is placed at any point on the line joining the foci, the edge being supposed clamped. In the present paper the author proposes to solve the problem in its most general form, *i.e.*, when the load is concentrated at any point on the upper face of the plate and the edge, as before, is clamped.

Let the bounding ellipse be given by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad a>b>0.$$

In terms of elliptic co-ordinates introduced by the transformation

$$z=x+iy=c\cosh(\xi+i\eta), \quad c>0$$

$\xi=\alpha$ $(\alpha>0)$ represents the bounding ellipse if

$$a=c\cosh\alpha \quad \text{and} \quad b=c\sinh\alpha.$$

Let the load be placed at the point whose elliptic co-ordinates are (γ, δ) where $0\leqq\gamma<\alpha$. The Cartesian co-ordinates of the loaded point are

$$x_1=c\cosh\gamma\cos\delta,$$

$$y_1=c\sinh\gamma\sin\delta.$$

Now, let (x, y) denote the Cartesian co-ordinates of any point on the mid-plane and put

$$r^2=(x-x_1)^2+(y-y_1)^2.$$

Further, let us assume for the deflexion the expression given by

$$w=-Wr^2(\log r-\tfrac{1}{2})/8\pi D+w' \tag{1}$$

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* The results of this paper were found more than three years ago. But the author was so long unable to establish the convergence of the series defining the deflexion. At the suggestion of Dr. A. C. Stevenson he again took up the problem and succeeded in establishing the convergence of the series and in settling other allied topics

where w' is some suitable continuous function of (x, y) biharmonic throughout the plate yielding continuous partial derivatives of at least the fourth order.

We may verify that, for $\xi > \gamma$ (*Cf.* Mathieu, 1869),

$$r^2(\log r - \tfrac{1}{2}) = \tfrac{1}{4}c^2 \sum_{n=0}^{\infty} \{P_n(\xi) \cos n\eta + Q_n(\xi) \sin n\eta\} \tag{2}$$

where

$$P_0(\xi) = \xi L_0(\xi) + L_0^{(1)} \exp(2\xi) + L_0^{(2)} + L_0^{(3)} \exp(-2\xi)$$

$$P_1(\xi) = \xi L_1(\xi) + L_1^{(1)} \exp(\xi) + L_1^{(2)} \exp(-\xi) + L_1^{(3)} \exp(-3\xi)$$

$$Q_1(\xi) = \xi M_1(\xi) + M_1^{(1)} \exp(\xi) + M_1^{(2)} \exp(-\xi) + M_1^{(3)} \exp(-3\xi)$$

$$P_2(\xi) = \xi L_2(\xi) + L_2^{(1)} + L_2^{(2)} \exp(-2\xi) + L_2^{(3)} \exp(-4\xi)$$

$$Q_2(\xi) = M_2^{(1)} + M_2^{(2)} \exp(-2\xi) + M_2^{(3)} \exp(-4\xi),$$

and for $n \geqq 3$

$$P_n(\xi) = L_n^{(1)} \exp\{-(n-2)\xi\} + L_n^{(2)} \exp\{-n\xi\} + L_n^{(3)} \exp\{-(n+2)\xi\}$$

$$Q_n(\xi) = M_n^{(1)} \exp\{-(n-2)\xi\} + M_n^{(2)} \exp\{-n\xi\} + M_n^{(3)} \exp\{-(n+2)\xi\}$$

where

$$L_0(\xi) = 2(\cosh 2\xi + \cosh 2\gamma + \cos 2\delta)$$

$$L_1(\xi) = -8 \cosh \xi \cosh \gamma \cos \delta$$

$$M_1(\xi) = -8 \sinh \xi \sinh \gamma \sin \delta$$

$$L_2(\xi) = 2$$

and

$$L_0^{(1)} = \log \tfrac{1}{2}c - \tfrac{1}{2}$$

$$L_0^{(2)} = (2 \log \tfrac{1}{2}c + 1)(\cosh 2\gamma + \cos 2\delta)$$

$$L_0^{(3)} = \log \tfrac{1}{2}c + \tfrac{3}{2} + \cosh 2\gamma \cos 2\delta$$

$$L_1^{(1)} = -4 \log \tfrac{1}{2}c . \cosh \gamma \cos \delta$$

$$L_1^{(2)} = -4(1 + \log \tfrac{1}{2}c) \cosh \gamma \cos \delta - (\cosh 3\gamma \cos \delta + \cosh \gamma \cos 3\delta)$$

$$L_1^{(3)} = \tfrac{1}{3} \cosh 3\gamma \cos 3\delta - \cosh \gamma \cos \delta$$

$$M_1^{(1)} = -4 \log \tfrac{1}{2}c . \sinh \gamma \sin \delta$$

$$M_1^{(2)} = 4(1 + \log \tfrac{1}{2}c) \sinh \gamma \sin \delta - (\sinh 3\gamma \sin \delta + \sinh \gamma \sin 3\delta)$$

$$M_1^{(3)} = \tfrac{1}{3} \sinh 3\gamma \sin 3\delta - \sinh \gamma \sin \delta$$

$$L_2^{(1)} = (1 + 2 \log \tfrac{1}{2}c) + \cosh 2\gamma \cos 2\delta$$

$$L_2^{(2)} = -\tfrac{1}{3}(\cosh 4\gamma \cos 2\delta + \cosh 2\gamma \cos 4\delta) + (\cosh 2\gamma + \cos 2\delta)$$

$$L_2^{(3)} = \tfrac{1}{6} \cosh 4\gamma \cos 4\delta - \tfrac{1}{3} \cosh 2\gamma \cos 2\delta$$

$$M_2^{(1)} = \sinh 2\gamma \sin 2\delta$$

$$M_2^{(2)} = -\tfrac{1}{3}(\sinh 4\gamma \sin 2\delta + \sinh 2\gamma \sin 4\delta)$$

$$M_2^{(3)} = \tfrac{1}{6} \sinh 4\gamma \sin 4\delta - \tfrac{1}{3} \sinh 2\gamma \sin 2\delta$$

$$L_n^{(1)} = \frac{2}{n(n-1)} \cosh n\gamma \cos n\delta - \frac{2}{(n-1)(n-2)} \cosh(n-2)\gamma \cos(n-2)\delta$$

$$L_n^{(2)} = -\frac{2}{n(n+1)}\{\cosh(n+2)\gamma \cos n\delta + \cosh n\gamma \cos(n+2)\delta\}$$
$$+\frac{2}{n(n-1)}\{\cosh n\gamma \cos(n-2)\delta + \cosh(n-2)\gamma \cos n\delta\}$$

$$L_n^{(3)} = \frac{2}{(n+1)(n+2)} \cosh(n+2)\gamma \cos(n+2)\delta - \frac{2}{n(n+1)} \cosh n\gamma \cos n\delta$$

$$M_n^{(1)} = \frac{2}{n(n-1)} \sinh n\gamma \sin n\delta - \frac{2}{(n-1)(n-2)} \sinh(n-2)\gamma \sin(n-2)\delta$$

$$M_n^{(2)} = -\frac{1}{n(n+1)}\{\sinh(n+2)\gamma \sin n\delta + \sinh n\gamma \sin(n+2)\delta\}$$
$$+\frac{2}{n(n-1)}\{\sinh n\gamma \sin(n-2)\delta + \sinh(n-2)\gamma \sin n\delta\}$$

$$M_n^{(3)} = \frac{2}{(n+1)(n+2)} \sinh(n+2)\gamma \sin(n+2)\delta - \frac{2}{n(n+1)} \sinh n\gamma \sin n\delta.$$

Let us assume (*Cf.* Timpe, 1923)

$$\begin{aligned} w' = {} & A_0' + A_0 \cosh 2\xi + (A_1' \cosh \xi + A_1 \cosh 3\xi) \cos \eta + (B_1' \sinh \xi + B_1 \sinh 3\xi) \sin \eta \\ & + \sum_{n=2}^{\infty} [\{A_{n-2} \cosh(n-2)\xi + A_n' \cosh n\xi + A_n \cosh(n+2)\xi\} \cos n\eta \\ & + \{B_{n-2} \sinh(n-2)\xi + B_n' \sinh n\xi + B_n \sinh(n+2)\xi\} \sin n\eta] \quad (3) \end{aligned}$$

where A's and B's are suitable constants to be determined from the boundary conditions.

It is evident that w is a formal solution of the differential equation

$$\nabla_1^4 w = 0$$

throughout the elliptic plate excepting the point (γ, δ). Also

$$\int_c N ds = -W$$

is formally satisfied where N is the shearing stress and the integral is taken over a closed curve lying entirely within the ellipse and enclosing the point (γ, δ) in its interior.

The boundary conditions for the clamped edge are

$$w = \partial w/\partial\xi = 0 \quad \text{over} \quad \xi = \alpha.$$

These give the following recurrence formulæ for the determination of the values of the constants A's and B's. They are

$$A_0' \sinh 2\alpha = \frac{Wc^2}{32\pi D}\,[(\cosh 2\alpha + \cosh 2\gamma + \cos 2\delta)(2 \sinh 2\alpha - \cosh 2\alpha) - \alpha \sinh 4\alpha - L_0^{(1)} + L_0^{(2)} \sinh 2\alpha + L_0^{(3)}] \quad (4)$$

$$A_0 \sinh 2\alpha = \frac{Wc^2}{32\pi D}\,[\cosh 2\alpha + \cosh 2\gamma + \cos 2\delta + 2\alpha \sinh 2\alpha + L_0^{(1)} \exp(2\alpha) - L_0^{(3)} \exp(-2\alpha)] \quad (5)$$

$$A_1'(\sinh 4\alpha + 2 \sinh 2\alpha) = \frac{Wc^2}{32\pi D}\,[2(\cosh 2\alpha + \cosh 2\gamma + \cos 2\delta)(3\alpha \sinh 2\alpha - \cosh 2\alpha) - 4\alpha \sinh 2\alpha \cosh 3\alpha + L_1^{(1)}(3 \sinh 3\alpha - \cosh 3\alpha) \exp(\alpha) + L_1^{(2)}(3 \sinh 3\alpha + \cosh 3\alpha) \exp(-\alpha) + 3L_1^{(3)}] \quad (6)$$

$$A_1(\sinh 4\alpha + 2 \sinh 2\alpha) = -\frac{Wc^2}{32\pi D}\,[2(\cosh 2\alpha + \cosh 2\gamma + \cos 2\delta)(\alpha \sinh \alpha - \cosh \alpha) - 2\alpha \sinh 4\alpha - L_1^{(1)} + L_1^{(2)} + L_1^{(3)}(3 \cosh \alpha + \sinh \alpha) \exp(-3\alpha)] \quad (7)$$

$$B_1'(\sinh 4\alpha - 2 \sinh 2\alpha) = -\frac{Wc^2}{32\pi D}\,[8 \sinh \alpha \cosh \gamma \cos \delta(3\alpha \cosh 3\alpha - \sinh 3\alpha) + 8\alpha \sinh 3\alpha \cosh \alpha \cosh \gamma \cos \delta - M_1^{(1)}(3 \cosh 3\alpha - \sinh 3\alpha) \exp(\alpha) - M_1^{(2)}(3 \cosh 3\alpha + \sinh 3\alpha) \exp(-\alpha) - 3M_1^{(3)}] \quad (8)$$

$$B_1(\sinh 4\alpha - 2 \sinh 2\alpha) = \frac{Wc^2}{32\pi D}\,[8 \sinh \alpha \sinh \gamma \sin \delta(\alpha \cosh \alpha - \sinh \alpha) - 4\alpha \sinh 2\alpha \sinh \gamma \sin \delta - M_1^{(1)} - M_1^{(2)} - M_1^{(3)}(\cosh \alpha - 3 \sinh \alpha) \exp(-3\alpha)] \quad (9)$$

$$A_2'(\sinh 6\alpha + 3 \sinh 2\alpha) = -4A_0 \sinh 4\alpha + \frac{Wc^2}{16\pi D}\,[(4\alpha \sinh 4\alpha - \cosh 4\alpha) + 2L_2^{(1)} \sinh 4\alpha + L_2^{(2)}(2 \sinh 4\alpha + \cosh 4\alpha) \exp(-2\alpha) + 2L_2^{(3)}] \quad (10)$$

$$A_2(\sinh 6\alpha + 3 \sinh 2\alpha) = 2A_0 \sinh 2\alpha - \frac{Wc^2}{32\pi D}\,[(4\alpha \sinh 2\alpha - 2 \cosh 2\alpha) + 2L_2^{(1)} \sinh 2\alpha + L_2^{(2)} + 2L_2^{(3)}(\sinh 2\alpha + 2 \cosh 2\alpha) \exp(-4\alpha)] \quad (11)$$

$$B_2'(\sinh 6\alpha - 3 \sinh 2\alpha) = \frac{Wc^2}{16\pi D}\,[2M_2^{(1)} \cosh 4\alpha + M_2^{(2)}(2 \cosh 4\alpha + \sinh 4\alpha)\exp(-2\alpha) + 2M_2^{(3)}] \quad (12)$$

$$B_2(\sinh 6\alpha - 3 \sinh 2\alpha) = -\frac{Wc^2}{16\pi D}\,[M_2^{(1)} \cosh 4\alpha + M_2^{(2)} + M_2^{(3)}(\cosh 2\alpha + 2 \sinh 2\alpha) \exp(-4\alpha)] \quad (13)$$

$$A_n'[\sinh(2n+2)\alpha + (n+1) \sinh 2\alpha] = -A_{n-2}(2 \sinh 2n\alpha + n \sinh 4\alpha) + \frac{Wc^2}{32\pi D}\,[L_n^{(1)}\{(n+2) \sinh(n+2)\alpha + (n-2) \cosh(n+2)\alpha\} \exp\{-(n-2)\alpha\} + L_n^{(2)}\{(n+2) \sinh(n+2)\alpha + n \cosh(n+2)\alpha\} \exp(-n\alpha) + (n+2)L_n^{(3)}] \quad (14)$$

$$A_n[\sinh(2n+2)\alpha+(n+1)\sinh 2\alpha] = A_{n-2}[(n-1)\sinh 2\alpha+\sinh(2n-2)\alpha]$$
$$-\frac{Wc^2}{32\pi D}\,[L_n^{(1)}\{n\sinh n\alpha+(n-2)\cosh n\alpha\}\exp\{-(n-2)\alpha\}+nL_n^{(2)}$$
$$+L_n^{(3)}\{n\sinh n\alpha+(n+2)\cosh n\alpha\}\exp\{-(n+2)\alpha\}] \quad (15)$$

$$B_n[\sinh(2n+2)\alpha-(n-1)\sinh 2\alpha] = -B_{n-2}[2\sinh 2n\alpha-n\sinh 4\alpha]$$
$$+\frac{Wc^2}{32\pi D}[M_n^{(1)}\{(n+2)\cosh(n+2)\alpha+(n-2)\sinh(n+2)\alpha\}\exp\{-(n-2)\alpha\}$$
$$+M_n^{(2)}\{(n+2)\cosh(n+2)\alpha+n\sinh n\alpha\}\exp\{-n\alpha\}+(n+2)M_n^{(3)}] \quad (16)$$

$$B_n[\sinh(2n+2)\alpha-(n+1)\sinh 2\alpha] = B_{n-2}[\sinh(2n-2)\alpha-(n-1)\sinh 2\alpha]$$
$$-\frac{Wc^2}{32\pi D}\,[M_n^{(1)}\{n\cosh n\alpha+(n-2)\sinh n\alpha\}\exp\{-(n-2)\alpha\}+nM_n^{(2)}$$
$$+M_n^{(3)}\{n\cosh n\alpha+(n+2)\sinh n\alpha\}\exp\{-(n+2)\alpha\}] \quad (17)$$

Now, writing

$$f_n = \sinh(2n+2)\alpha+(n+1)\sinh 2\alpha$$

we have

$$A_nf_n = A_{n-2}f_{n-2}-\frac{Wc^2}{32\pi D}\,[L_n^{(1)}\{(n-1)-\exp(-2n\alpha)\}\exp(2\alpha)$$
$$+nL_n^{(2)}+L_n^{(3)}\{(n+1)+\exp(-2n\alpha)\}\exp(-2\alpha)] \quad (18)$$

Putting $n = 4, 6, 8, \ldots, 2m$ successively in the above and adding, we get

$$A_{2m}f_{2m} = A_2f_2-\frac{Wc^2}{32\pi D}\,[\{3L_4^{(1)}+5L_6^{(1)}+7L_8^{(1)}+\cdots+(2m-1)L_{2m}^{(1)}\}\exp(4\alpha)$$
$$-\{L_4^{(1)}\exp(-6\alpha)+L_6^{(1)}\exp(-10\alpha)+L_8^{(1)}\exp(-14\alpha)+\cdots+L_{2m}^{(1)}\exp\{-(4m-2)\alpha\}\}$$
$$+4L_4^{(2)}+6L_6^{(2)}+8L_8^{(2)}+\cdots+2mL_{2m}^{(2)}+\{5L_4^{(3)}+7L_6^{(3)}+9L_8^{(3)}+\cdots+(2m+1)L_{2m}^{(3)}\}\exp(-2\alpha)$$
$$+\{L_4^{(3)}\exp(-10\alpha)+L_6^{(3)}\exp(-14\alpha)+L_8^{(3)}\exp(-18\alpha)+\cdots+L_{2m}^{(3)}\exp\{-(4m+2)\alpha\}\}] \quad (19)$$

Similarly, we can easily show that

$$A_{2m+1}f_{2m+1} = A_1f_1-\frac{Wc^2}{32\pi D}\,[\{2L_3^{(1)}+4L_5^{(1)}+6L_7^{(1)}+\cdots+2nL_{2n+1}^{(1)}\}\exp(2\alpha)$$
$$-\{L_3^{(1)}\exp(-4\alpha)+L_5^{(1)}\exp(-8\alpha)+L_7^{(1)}\exp(-12\alpha)+\cdots+L_{2n+1}^{(1)}\exp(-4n\alpha)\}$$
$$+3L_3^{(2)}+5L_5^{(2)}+7L_7^{(2)}+\cdots+(2n+1)L_{2n+1}^{(2)}$$
$$+\{4L_3^{(3)}+6L_5^{(3)}+8L_7^{(3)}+\cdots+(2n+2)L_{2n+1}^{(3)}\}\exp(-2\alpha)$$
$$+L_3^{(3)}\exp(-8\alpha)+L_5^{(3)}\exp(-12\alpha)+L_7^{(3)}\exp(-16\alpha)+\cdots+L_{2n+1}^{(3)}\exp\{-(4n+4)\alpha\}] \quad (20)$$

Also writing,

$$g_n = \sinh(2n+2)\alpha-(n+1)\sinh 2\alpha$$

we have

$$B_{2n}g_{2n} = B_2g_2-\frac{Wc^3}{32\pi D}[\{3M_4^{(1)}+5M_6^{(1)}+7M_8^{(1)}+\cdots+(2n-1)M_{2n}^{(1)}\}\exp(2\alpha)$$
$$+\{M_4^{(1)}\exp(-6\alpha)+M_6^{(1)}\exp(-10\alpha)+M_8^{(1)}\exp(-14\alpha)+\cdots+M_{2n}^{(1)}\exp\{-(4n-2)\alpha\}\}$$
$$+\{4M_4^{(2)}+6M_6^{(2)}+8M_8^{(2)}+\cdots+2nM_{2n}^{(2)}\}+\{5M_4^{(3)}+7M_6^{(3)}+9M_8^{(3)}+\cdots+(2n+1)M_{2n}^{(3)}\}$$
$$+\{M_4^{(3)}\exp(-10\alpha)+M_6^{(3)}\exp(-14\alpha)+M_8^{(3)}\exp(-18\alpha)+\cdots+M_{2n}^{(3)}\exp\{-(4n+2)\alpha\}] \quad (21)$$

and

$$B_{2n+1}g_{2n+1} = B_1g_1-\frac{Wc^2}{32\pi D}[\{2M_3^{(1)}+4M_5^{(1)}+6M_7^{(1)}+\cdots+2nM_{2n+1}^{(1)}\}\exp(2\alpha)$$
$$+\{M_3^{(1)}\exp(-4\alpha)+M_5^{(1)}\exp(-8\alpha)+M_7^{(1)}\exp(-12\alpha)+\cdots+M_{2n+1}^{(1)}\exp(-4n\alpha)\}$$
$$+3M_3^{(2)}+5M_5^{(2)}+7M_7^{(2)}+\cdots+(2n+1)M_{2n+1}^{(2)}$$
$$+\{4M_3^{(3)}+6M_5^{(3)}+8M_7^{(3)}+\cdots+(2n+2)M_{2n+1}^{(3)}\}\exp(-2\alpha)$$
$$-\{M_3^{(3)}\exp(-8\alpha)+M_5^{(3)}\exp(-12\alpha)+M_7^{(3)}\exp(-16\alpha)+\cdots+M_{2n+1}^{(3)}\exp\{-4(n+1)\alpha\}\}] \quad (22)$$

We may now easily show that, for $n \geqq 3$

$$|L_n^{(1)}| \leqq \frac{4}{n(n-2)}\cosh n\gamma, \qquad |M_n^{(1)}| \leqq \frac{4}{n(n-2)}\sinh n\gamma$$

$$|L_n^{(2)}| \leqq \frac{8}{(n^2-1)}\cosh(n+2)\gamma, \qquad |M_n^{(2)}| \leqq \frac{8}{(n^2-1)}\sinh(n+2)\gamma.$$

$$|L_n^{(3)}| \leqq \frac{4}{n(n+2)}\cosh(n+2)\gamma, \qquad |M_n^{(3)}| \leqq \frac{4}{n(n+2)}\sinh(n+2)\gamma.$$

Since,

$$L_n^{(1)} = \frac{2}{n(n-1)}\cosh n\gamma\cos n\delta-\frac{2}{(n-1)(n-2)}\cosh(n-2)\gamma\cos(n-2)\delta$$

we easily prove,

$$3L_4^{(1)}+5L_6^{(1)}+7L_8^{(1)}+\cdots+(2n-1)L_{2n}^{(1)} = \frac{1}{n}\cosh 2n\gamma\cos 2n\delta-\cosh 2\gamma\cos 2\delta.$$

Therefore

$$|3L_4^{(1)}+5L_6^{(1)}+7L_8^{(1)}+\cdots+(2n-1)L_{2n}^{(1)}| < 2\cosh 2n\gamma \quad \text{for} \quad n \geqq 2.$$

Again, it is easy to shew that

$$|L_4^{(1)}\exp(-6\alpha)+L_6^{(1)}\exp(-10\alpha)+L_8^{(1)}\exp(-14\alpha)+\cdots+L_{2n}^{(1)}\exp\{-(4n-2)\alpha\}| < \cosh 2n\gamma$$

It is also easy to shew that

$$|4L_4^{(2)}+6L_6^{(2)}+8L_8^{(2)}+\cdots+2nL_{2n}^{(2)}| \leqq 4\cosh(2n+2)\gamma \quad \text{for} \quad n \geqq 2.$$

And

$$|5L_4^{(3)}+7L_6^{(3)}+9L_8^{(3)}+\cdots+(2n+1)L_{2n}^{(3)}| < 2\cosh(2n+2)\gamma \quad \text{for} \quad n \geqq 2$$

and

$$|L_4^{(3)}\exp(-10\alpha)+L_6^{(3)}\exp(-14\alpha)+\cdots+L_{2n}^{(3)}\exp\{-(4n+2)\alpha\}| < \tfrac{1}{4}\cosh(2n+2)\gamma.$$

From (19), it easily follows that

$$|A_{2n} f_{2n}| < K \cosh(2n+2)\gamma,$$

where n is sufficiently large, and K is a positive number independent of n. We may similarly shew that

$$|A_{2n+1} f_{2n+1}| < K' \cosh(2n+3)\gamma,$$

where n is sufficiently large, and K' is a positive number independent of n.

It is now easy to shew that

$$|A_n| < D_1 \exp\{-n(2\alpha-\gamma)\},$$

where n is sufficiently large and D_1 is a positive number independent of n. We may also easily deduce that

$$|A_n'| < D_2 \exp\{-n(2\alpha-\gamma)\},$$

where n is sufficiently large and D_2 is a positive number independent of n.

We may easily shew that

$$3M_4^{(1)} + 5M_6^{(1)} + 7M_8^{(1)} + \cdots + (2n-1)M_{2n}^{(1)} = \frac{1}{n} \sinh 2n\gamma \sin 2n\delta - \sinh 2\gamma \sin 2\delta.$$

Therefore

$$|3M_4^{(1)} + 5M_6^{(1)} + 7M_8^{(1)} + \cdots + (2n-1)M_{2n}^{(1)}| \leqq 2 \sinh 2n\gamma.$$

Also

$$|M_4^{(1)} \exp(-6\alpha) + M_6^{(1)} \exp(-10\alpha) + \cdots + M_{2n}^{(1)} \exp\{-(2n-2)\alpha\}| < \sinh 2n\gamma.$$

We may easily shew that

$$|4M_4^{(2)} + 6M_6^{(2)} + 8M_8^{(2)} + \cdots + 2nM_n^{(2)}| < 4 \sinh(2n+2)\gamma$$

and

$$|5M_4^{(3)} + 7M_6^{(3)} + 9M_8^{(3)} + \cdots + (2n+1)M_{2n}^{(3)}| < 2 \sinh(2n+2)\gamma$$

and

$$|M_4^{(3)} \exp(-10\alpha) + M_6^{(3)} \exp(-14\alpha) + M_8^{(3)} \exp(-18\alpha) + \cdots + M_{2n}^{(3)} \exp\{-(4n+2)\alpha\}| < \tfrac{1}{2} \sinh(2n+2)\gamma.$$

These inequalities easily lead to the following results

$$|B_n| < D_3 \exp\{-n(2\alpha-\gamma)\}$$

and

$$|B_n'| < D_4 \exp\{-n(2\alpha-\gamma)\}$$

where n is sufficiently large and D_3 and D_4 are two positive numbers independent of n.

Turning back to the series (3) we split it up into six sub-series of the type

$$\sum A_n \cosh(n+2)\xi \cos n\eta.$$

Since, for sufficiently large n

$$|A_n \cosh(n+2)\xi \cos n\eta| < D_1' \exp\{-n(2\alpha-\gamma-\xi)\}$$

where $D_1' = D_1 \exp(2\alpha)$, and further since $2\alpha-\gamma-\xi > 0$, for all ξ in the interval $0 \leqq \xi \leqq \alpha$ so long as $0 \leqq \gamma < \alpha$, we conclude that the series

$$\sum A_n \cosh(n+2)\xi \cos n\eta$$

converges absolutely throughout the interior as well as the boundary of the ellipse $\xi = \alpha$. In fact, the series converges absolutely for all η and for all ξ for which $0 \leqq \xi < 2\alpha - \gamma$. But $2\alpha - \gamma > \alpha$ for $0 \leqq \gamma < \alpha$. This shows that the range of absolute convergence extends beyond the given ellipse. We can also easily shew that the series is uniformly convergent in both the variables in a closed bigger ellipse which contains the given ellipse with its boundary in its interior.

The same is also true of the other five series. It follows that the series given by formula (3) represents a function continuous throughout the plate including its boundary. The validity of the process employed for the determination of the recurrence formulæ which gives the constants A_n, A'_n, B_n, B'_n can be demonstrated without much difficulty.

On the other hand, starting with the fundamental transformation

$$z = x + iy = c\cosh(\xi + i\eta) = c(\cosh\xi \cos\eta + i\sinh\xi \sin\eta), \quad c > 0$$

we have

$$2c^2(\cosh 2\xi \cos 2\eta + i \sinh 2\xi \sin 2\eta) = (2z)^2 - \frac{2}{1!}c^2$$

$$2c^3(\cosh 3\xi \cos 3\eta + i\sinh 3\xi \sin 3\eta) = (2z)^3 - \frac{3}{1!}c^2(2z)$$

$$2c^4(\cosh 4\xi \cos 4\eta + i\sinh 4\xi \sin 4\eta) = (2z)^4 - \frac{4}{1!}c^2(2z)^2 + \frac{4.1}{2!}c^4.$$

In general

$$2c^n(\cosh n\xi \cos n\eta + i\sinh n\xi \sin n\eta) = (2z)^n - \frac{n}{1!}c^2(2z)^{n-2} + \frac{n(n-3)}{2!}c^4(2z)^{n-4}$$

$$-\frac{n(n-4)(n-5)}{3!}c^6(2z)^{n-6} + \cdots + (-)^r \frac{n(n-r-1)(n-r-2)\cdots(n-2r+1)}{r!}c^{2r}(2z)^{n-2r} + \cdots$$

We shall denote the above by the symbol $P_n(z)$. On the other hand

$$c^2(\cosh 2\xi + \cos 2\eta) = 2(x^2 + y^2) = 2z\bar{z}$$

$$2c^3\{(\cosh 3\xi \cos\eta + \cosh\xi \cos 3\eta) + i(\sinh 3\xi \sin\eta + \sinh\xi \sin 3\eta)\} = 2\bar{z}P_2(z)$$

$$2c^4\{(\cosh 4\xi \cos 2\eta + \cosh 2\xi \cos 4\eta) + i(\sinh 4\xi \sin 2\eta + \sinh 2\xi \sin 4\eta)\} = 2\bar{z}P_3(z)$$

And in general

$$2c^n[\{\cosh n\xi \cos(n-2)\eta + \cosh(n-2)\xi \cos n\eta\}$$
$$+ i\{\sinh n\xi \sin(n-2)\eta + \sinh(n-2)\xi \sin n\eta\}] = 2\bar{z}P_{n-1}(z).$$

It is clear, that

$$\sum_{n=0}^{\infty} A'_n \cosh n\xi \cos n\eta$$

is the real part of the series

See Hobson—Plane Trigonometry, 3rd edition, p. 105.

$$A_0' + \sum_{n=1}^{\infty} \frac{A_n'}{2c^n} P_n(z).$$

The above series has been proved to be uniformly convergent in any closed domain lying entirely within the ellipse $\xi = 2\alpha - \gamma$. So the sum represents an analytic function in the interior of the ellipse $\xi = 2\alpha - \gamma$. Denoting it by $u(x, y) + iv(x, y)$, we have

$$u(x, y) = \sum_{n=0}^{\infty} A_n' \cosh n\xi \cos n\eta.$$

$u(x, y)$ being the real part of an analytic function is plane harmonic within the ellipse $\xi = 2\alpha - \gamma$, and hence throughout the mid plane of the plate $\nabla_1^2 u = 0$ and so certainly $\nabla_1^4 u = 0$, there.

Again

$$\sum B_1' \sinh n\xi \sin n\eta$$

is the imaginary part of the series

$$\sum_{n=1}^{\infty} \frac{B_n'}{2c^n} P_n(z).$$

Denoting the sum of the above series by $u' + iv'$, we have

$$v' = \sum B_1' \sinh n\xi \sin n\eta.$$

Evidently $\nabla_1^2 v' = 0$, throughout the interior of the ellipse $\xi = 2\alpha - \gamma$. So certainly $\nabla_1^4 v' = 0$ throughout the mid-plane of the plate.

The contributions to the stress-resultant N and the stress-couples G and H due to these, *viz.*, u and v' are finite and continuous throughout the plate including the boundary. The series

$$U = A_0 \cosh 2\xi + A_1 \cosh 3\xi \cos \eta + \sum_{n=2}^{\infty} \{A_{n-2} \cosh(n-2)\xi + A_n \cosh(n+2)\xi\} \cos n\eta$$

is absolutely convergent throughout the interior of the ellipse $\xi = 2\alpha - \gamma$. So we may re-arrange it in any way we like and write

$$U = \sum_{n=0}^{\infty} \{A_n \cosh(n+2)\xi \cos n\eta + \cosh n\xi \cos(n+2)\eta\}.$$

Evidently

$$U = \text{Real part of } \frac{\bar{z}}{c}\left[\frac{2A_0 z}{c} + \sum_{n=1}^{\infty} \frac{A_n}{c^{n+1}} P_{n+1}(z)\right]$$

The series within the brackets is uniformly convergent in any closed domain lying entirely within the interior of the ellipse $\xi = 2\alpha - \gamma$. It therefore represents an analytic function in the above open domain. Denoting it by $u'' + iv''$, we have $U = xu'' + yv''$. It is easily seen that $\nabla_1^4 U = 0$ throughout the interior of the above ellipse. Similar remark applies to the series

$$V = B_1 \sinh 3\xi \sin \eta + \sum_{n=2}^{\infty} \{B_{n-2} \sinh(n-2)\xi + B_n \sinh(n+2)\xi\} \sin n\eta.$$

The contributions to the stress-resultant N and the stress-couples G and H due to these functions may easily be shewn to be finite and continuous throughout the elliptic plate including the boundary.

It is easy to shew that

$$\int_c N ds = -W,$$

where c is any closed curve lying entirely within the plate and enclosing the point (γ, δ) in its interior, while

$$\int_\Gamma N ds = 0,$$

where Γ is a closed curve lying entirely within the elliptic plate, and which does not contain the point (γ, δ) in its interior or on its boundary.

It therefore follows that the function w defined by

$$w = -Wr^2(\log r - \tfrac{1}{2})/8\pi D + w'$$

where w' is given by (3) represents the deflexion of the central plane of an elliptic plate $0 \leqq \xi \leqq \alpha$ made of isotropic elastic material, clamped at the edge carrying a load of weight W concentrated at any point (γ, δ) on the upper face of the plate.

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ON LAPLACE TRANSFORM OF TWO VARIABLES

By

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1. In this paper I have added some more formulae to the well known Laplace transform of two variables.

2. **Theorem I.** *If*

$$\varphi_1(p, q) \doteqdot h_1(x, y) \text{ and } \varphi_2(p, q) \doteqdot h_2(x, y)$$

then

$$\int_0^\infty\!\!\int \varphi_1(u, v)h_2(u, v)\,\frac{du}{u}\,\frac{dv}{v} = \int_0^\infty\!\!\int h_1(t, s)\varphi_2(t, s)\,\frac{dt}{t}\,\frac{ds}{s} \tag{1}$$

provided that the order of integration may be inverted.

Proof : Here

$$\varphi_1(u, v) = uv\int_0^\infty\!\!\int e^{-ut-vs}h_1(t, s)dtds \tag{2}$$

and

$$\varphi_2(t, s) = ts\int_0^\infty\!\!\int e^{-tu-sv}h_2(u, v)dudv \tag{3}$$

and therefore

$$\begin{aligned}\int_0^\infty\!\!\int \varphi_1(u, v)h_2(u, v)\,\frac{du}{u}\,\frac{dv}{v} &= \int_0^\infty\!\!\int h_2(u, v)dudv\int_0^\infty\!\!\int e^{-ut-vs}h_1(t, s)dtds\\ &= \int_0^\infty\!\!\int h_1(t, s)\Big[\int_0^\infty\!\!\int e^{-ut-vs}h_2(u, v)dudv\Big]dsdt\\ &= \int_0^\infty\!\!\int h_1(t, s)\varphi_2(t, s)\frac{ds}{s}\,\frac{dt}{t}\end{aligned}$$

provided the change in the order of integration is permitted.

Regarding this change we note that the condition for the absolute and uniform convergence of (2) is $\mathbf{R}(\mu_1, \mu_2) > 0$ where $h_1(t, s) = O(t^{\mu_1})$ for small t and $h_1(t, s) = O(s^{\mu_2})$ for small s, an additional condition, if required, $h_1(t, s) = O\{\exp(-t^{\lambda_1})\}$, $\mathbf{R}(\lambda_1) > 0$ for large t and $h_1(t, s) = O\{\exp(-s^{\lambda_2})\}$, $\mathbf{R}(\lambda_2) > 0$ for large s. Integral (3) will be absolutely and uniformly convergent if $R(\mu_3, \mu_4) > 0$ where $h_2(t, s) = O(t^{\mu_3})$ for small t and $h_2(t, s) = O(s^{\mu_4})$ for small s, and an additional condition, if required, $h_2(t, s) = O\{\exp(-t^{\lambda_3})\}$, $\mathbf{R}(\lambda_3) > 0$ for large values of t and $h_2(t, s) = O\{\exp(-s^{\lambda_4})\}$, $\mathbf{R}(\lambda_4) > 0$ for large values of s.

We have imposed, however, hard conditions on $h_1(s, t)$ and $h_2(s, t)$ for the justification in the change of the order of integration. These conditions may be relaxed.

3. Theorem II. *If*

$$\varphi_n(p, q) \doteqdot (xy)^{-n-1} f(x, y),$$

where $f(x, y)$ *is not a function of* n, *then*

$$\varphi_{n+m}(p, q) = \frac{pq}{\Gamma(m)\Gamma(m)} \int\!\!\int_0^\infty (st)^{m-1} \frac{\varphi_n(p+s, q+t)}{(p+s)(q+t)}\, ds\, dt \tag{4}$$

provided $\mathbf{R}(p, q) > 0$, $\mathbf{R}(m) > 0$, $\mathbf{R}(\mu_\lambda - n) > 0$, $\lambda = 1, 2$, *where*

$$f(x, y) = O(x^{\mu_1}) \text{ for small values of } x,$$
$$= O(y^{\mu_2}) \text{ for small values of } y,$$

and the integral converges.

Proof: Since

$$\varphi_n(p, q) = pq \int\!\!\int_0^\infty e^{-px-qy}(xy)^{-n-1} f(x, y)\, dx\, dy, \tag{1}$$

$$\varphi_{n+m}(p, q) = pq \int\!\!\int_0^\infty e^{-px-qy}(xy)^{-m-n-1} f(x, y)\, dx\, dy, \tag{2}$$

$$x^{-m} = \frac{1}{\Gamma(m)} \int_0^\infty e^{-sx} s^{m-1}\, ds,$$

$$y^{-m} = \frac{1}{\Gamma(m)} \int_0^\infty e^{-ty} t^{m-1}\, dt, \quad \mathbf{R}(m) > 0.$$

Substituting in (2), we get

$$\varphi_{n+m}(p, q) = \int\!\!\int_0^\infty e^{-px-qy} f(x, y)(xy)^{-n-1}\, dx\, dy$$
$$\times \frac{1}{\Gamma(m)} \int_0^\infty e^{-sx} s^{m-1}\, ds \, \frac{1}{\Gamma(m)} \int_0^\infty e^{-ty} t^{m-1}\, dt \tag{3}$$

provided $\mathbf{R}(p, q) > 0$, $\mathbf{R}(m) \geq 0$, $\mathbf{R}(\mu_\lambda - n) > 0$, $\lambda = 1, 2$ and the integral converges. Hence we get the required result.

In order to justify the change of order of integration, we proceed as follows:

(*i*) We find that in (3) the s-integral is absolutely and uniformly convergent for $\mathbf{R}(m) > 0$ and the x-integral is absolutely and uniformly convergent for $\mathbf{R}(p) > 0$ and $\mathbf{R}(\mu_1 - n) \geq 0$, where $f(x, y) = O(x^{\mu_1})$ for small values of x, therefore the s-integral and x-integral are inter-changeable. Similarly, the t-integral and y-integral are interchangeable for $\mathbf{R}(q) > 0$ and $\mathbf{R}(\mu_2 - n) > 0$ where $f(x, y) = O(y^{\mu_2})$ for small values of y.

(*ii*) We may justify the change in another way, if $f(x, y)$ is of the form $f_1(x).f_2(y)$. Let

$$\theta(s) = s^{m-1} \int_0^\infty e^{-px-sx} f_1(x) x^{-n-1}\, dx$$

It is evident that $\theta(s)$ is uniformly convergent in $s \geq 0$, if $\mathbf{R}(m) > 0$ and $\mathbf{R}(\mu_1 - n) > 0$, where $f_1(x) \doteq O(x^{\mu_1})$ for small values of x.

Again, consider the integral

$$I = \int_0^\infty s^{m-1}\left[\int_A^\infty e^{-px-sx} f_1(x) x^{-n-1} dx\right] ds$$

where A is large.

Now, for large values of x, I does not exceed a constant multiple of

$$\int_0^\infty |s^{m-1}| \left[\int_A^\infty |e^{-px-sx} f_1(x) x^{-n-1}| dx\right] ds$$

which tends to zero, if $\mathbf{R}(p) > 0$, and the integral

$$\int_0^\infty |s^{m-1}| e^{-sx} ds$$

is convergent.

Similarly, we can show that t-integral and y-integral are inter-changeable for $\mathbf{R}(q) > 0$ and $\mathbf{R}(\mu_2 - n) > 0$ where $f_2(y) = O(y^{\mu_2})$ for small values of y.

4. Theorem III. *If*

$$f(p,\, q) \doteqdot h(x,\, y)$$

then

$$\frac{f(\log p,\, \log q)}{\log p \,.\, \log q} \doteqdot \iint_0^\infty \frac{x^s y^t h(s,\, t)}{\Gamma(s+1)\Gamma(t+1)} ds\, dt \tag{1}$$

Proof: We have

$$f(p,\, q) \doteqdot f(x,\, y).$$

On replacing p and q by $\log p$ and $\log q$ respectively, we get

$$f(\log p,\, \log q) = \log p \,.\, \log q \iint_0^\infty p^{-s} q^{-t} h(s,\, t)\, ds\, dt. \tag{2}$$

But we know that

$$p^{-s} \doteqdot \frac{x^s}{\Gamma(s+1)}, \quad q^{-t} \doteqdot \frac{y^t}{\Gamma(t+1)}, \quad \mathbf{R}(p,\, q) > 0.$$

Hence, from (2), we obtain

$$\frac{f(\log p,\, \log q)}{\log p \,.\, \log q} \doteqdot \iint_0^\infty \frac{x^s y^t h(s,\, t)}{\Gamma(s+1)\Gamma(t+1)} ds\, dt$$

provided the integral converges.

5. Theorem IV. *If*

$$f(p,\, q) \doteqdot h(x,\, y)$$

then

$$f\left(\frac{1}{p^2}, \frac{1}{q^2}\right) \doteqdot \tfrac{1}{4}x^2y^2 \iint\limits_0^\infty {}_0F_2[\tfrac{3}{2}, 2; -\tfrac{1}{4}sx^2]. \, {}_0F_2[\tfrac{3}{2}, 2; -\tfrac{1}{4}ty^2]\, h(s, t)\,ds\,dt \tag{1}$$

provided the integral converges.

Proof: We have

$$f(p, q) \doteqdot h(x, y)$$

and so

$$f\left(\frac{1}{p^2}, \frac{1}{q^2}\right) = \iint\limits_0^\infty \frac{e^{-s/p^2}e^{-t/q^2}}{p^2q^2} h(s, t)\,ds\,dt. \tag{2}$$

We can easily find the operational of $p^2 \exp(-sp^{-2})$. This function may be written as

$$\sum_{n=0}^{\infty} \frac{(-s)^n}{n!\,p^{2n+2}}$$

and the original is

$$\sum_{n=0}^{\infty} \frac{(-s)^n x^{2n+2}}{n!\,(2n+2)!}.$$

Further, we can write this in the hypergeometric form

$$\tfrac{1}{2}x^2 . {}_0F_2[\tfrac{3}{2}, 2; -\tfrac{1}{4}sx^2].$$

Hence from (2), we obtain

$$f\left(\frac{1}{p^2}, \frac{1}{q^2}\right) \doteqdot \tfrac{1}{4}x^2y^2 \iint\limits_0^\infty {}_0F_2[\tfrac{3}{2}, 2; -\tfrac{1}{4}sx^2]. {}_0F_2[\tfrac{3}{2}, 2; -\tfrac{1}{4}ty^2]\, h(s, t)\,ds\,dt$$

This may be further reduced by

$$J_{m,n}(x) = \frac{x^{m+n}}{3^{m+n}\Gamma(m+1)\Gamma(n+1)} \cdot {}_0F_2\left[m+1, n+1; -\frac{x^3}{27}\right]$$

to

$$f\left(\frac{1}{p^2}, \frac{1}{q^2}\right) \doteqdot \frac{xy\pi}{4} \iint\limits_0^\infty J_{1,\frac{1}{2}}(3[\tfrac{1}{4}sx^2]^{\frac{1}{3}}).J_{1,\frac{1}{2}}(3[\tfrac{1}{4}ty^2]^{\frac{1}{3}}). \frac{h(s, t)}{(st)^{\frac{1}{2}}}\,ds\,dt.$$

6. Theorem V. *If*

$$f(p, q) \doteqdot h(x, y)$$

and

$$h(p, q) \doteqdot g(x, y)$$

then

$$f(p, q) = pq \iint\limits_0^\infty \frac{g(s, t)}{(p+s)^2(q+t)^2}\,ds\,dt \tag{1}$$

provided the integral is convergent.

Proof: We have

$$f(p, q) = pq \iint\limits_0^\infty e^{-px-qy} h(x, y)\,dx\,dy \tag{2}$$

$$h(x, y) = xy \iint\limits_0^\infty e^{-xs-yt} g(s, t)\,ds\,dt \tag{3}$$

Hence

$$f(p, q) = pq \int\!\!\int_0^\infty e^{-px-qy} dxdy \left[xy \int\!\!\int_0^\infty e^{-xs-yt} g(s, t) ds dt \right]$$

$$= pq \int\!\!\int_0^\infty \frac{g(s, t)}{(s+p)^2(t+q)^2} ds dt. \tag{4}$$

provided the change of order of integration is permissible.

Regarding the change of order of integration, (2) is absolutely and uniformly convergent if $\mathbf{R}(p) \geq 0$, $\mathbf{R}(q) > 0$ and $\mathbf{R}(\mu_1) > 0$, $\mathbf{R}(\mu_2) > 0$ where

$$h(x, y) = O(x^{\mu_1}) \text{ for small } x,$$
$$= O(y^{\mu_2}) \text{ for small } y,$$

and integral (3) is absolutely and uniformly convergent if $\mathbf{R}(x) > 0$, $\mathbf{R}(y) > 0$ and $\mathbf{R}(\nu_1) > 0$, $\mathbf{R}(\nu_2) > 0$ where $g(s, t) = O(s^{\nu_1})$ for small s and $g(s, t) = O(t^{\nu_2})$ for small t and the integral (4) is convergent.

7. **Theorem VI.** *If*

$$\varphi(p, q) \doteqdot f(x, y)$$

and

$$\sqrt{(pq)}.h\left(\frac{1}{p}, \frac{1}{q}\right) \doteqdot g(x, y),$$

then

$$\varphi(p^2, q^2) \doteqdot \tfrac{1}{4}\pi xy g(\tfrac{1}{4}x^2, \tfrac{1}{4}y^2). \tag{1}$$

Proof: We have

$$\varphi(p, q) = pq \int\!\!\int_0^\infty e^{-pu-qv} h(u, v) du dv$$

$$= pq \int\!\!\int_0 e^{-p/u-q/v} \sqrt{(uv)}.h\left(\frac{1}{u}, \frac{1}{v}\right) \frac{du}{u^2\sqrt{u}} . \frac{dv}{v^2\sqrt{v}}. \tag{2}$$

Since

$$\sqrt{(pq)}.h\left(\frac{1}{p}, \frac{1}{q}\right) \doteqdot g(x, y) \tag{3}$$

so, (2) takes the form

$$\frac{\varphi(p, q)}{pq} = \int\!\!\int_0^\infty e^{-p/u-q/v} \left[uv \int\!\!\int_0^\infty e^{-ux-vy} g(x, y) dxdy \right] \frac{du}{u^2\sqrt{u}} . \frac{dv}{v^2\sqrt{v}}$$

$$= \int\!\!\int_0^\infty g(x, y) \left[\int\!\!\int_0^\infty e^{-p/u-ux} e^{-q/v-vy} \frac{du}{u\sqrt{u}} . \frac{dv}{v\sqrt{v}} \right] dxdy \tag{4}$$

on changing the order of integration. We know that

$$\int_0^\infty e^{-p/u-ux} \frac{du}{u\sqrt{u}} = (\pi/p)^{\frac{1}{2}}. \exp\left[-2\sqrt{(px)}\right]$$

therefore (3), reduces to

$$\varphi(p, q) = \pi\sqrt{(pq)} \int\!\!\int_0^\infty g(x, y) \exp\left[-2(px)^{\frac{1}{2}} - 2(qy)^{\frac{1}{2}}\right] dxdy. \tag{5}$$

Hence
$$\varphi(p^2, q^2) \doteqdot \tfrac{1}{4}\pi xyg(\tfrac{1}{4}x^2, \tfrac{1}{4}y^2).$$

Regarding the change of order of integration in (4), we find that (2) is uniformly and absolutely convergent with respect to p and q, if $e^{-p_0u}h(u, v)$ and $e^{-q_0v}h(u, v)$ are bounded for $u \geq 0$ and $v \geq 0$, also for $\mathbf{R}(p) > \mathbf{R}(p_0) > 0$ and $\mathbf{R}(q) > \mathbf{R}(q_0) > 0$. Integral (3) is uniformly and absolutely convergent with respect to u and v, if $e^{-u_0x}g(x, y)$ and $e^{-v_0y}g(x, y)$ are bounded for $x \geq 0$ and $y \geq 0$ for $\mathbf{R}(u) > \mathbf{R}(u_0) > 0$ and $\mathbf{R}(v) > \mathbf{R}(v_0) > 0$. Further, integral (5: is convergent.

8. **Theorem VII.** *(a) If*
$$\varphi(p, q) \doteqdot f(x, y),$$
then
$$e^{-\lambda p-\mu q}\varphi(p, q) \doteqdot 0, \; x < \lambda, \; y < \mu \tag{1}$$
$$e^{-\lambda p-\mu q}\varphi(p, q) \doteqdot f(x-\lambda, y-\mu), \; x > \lambda, \; y > \mu,$$
where λ *and* μ *are two positive numbers and* $f(x, y) = 0$ *for* $x < 0$ *and* $y < 0$.

We may write
$$e^{-\lambda p-\mu q}\varphi(p, q) = pq \iint\limits_0^\infty e^{-p(\lambda+x)-q(\mu+y)}f(x, y)dxdy$$
$$= pq\int_\lambda^\infty \int_\mu^\infty e^{-px-qy}f(x-\lambda, y-\mu)dxdy,$$
whence, by the hypothesis, we obtain (1).

(b) If
$$\varphi(p, q) \doteqdot f(x, y),$$
then
$$e^{\lambda p+\mu q}\varphi(p, q) \doteqdot f(x+\lambda, y+\mu) \tag{2}$$
where $f(x, y) = 0$ *for* $0 < x < \lambda$ *and* $0 < y < \mu$ *and* λ *and* μ *are any two positive numbers.*

We can write
$$e^{\lambda p+\mu q}\varphi(p, q) = pq \iint\limits_0^\infty e^{-p(x-\lambda)-q(y-\mu)}f(x, y)dxdy$$
$$= pq\int_{-\lambda}^\infty \int_{-\mu}^\infty e^{-px-qy}f(x+\lambda, y+\mu)dxdy$$
$$= pq\int_0^\infty \int_0^\infty e^{-pu-qv}f(u+\lambda, v+\mu)dudv,$$
whence, we obtain (2).

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DEVELOPMENT OF LIQUID MOTION DUE TO AN IMPULSE

By

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We assume that at time $t = 0$, suddenly an impulse is applied in a plane to an infinite liquid otherwise at rest. To such a problem where no wall is present, the approximations of the boundary-layer theory hold good. For motion in two-dimensions, we take the x-axis along the given straight line and the y-axis perpendicular thereto. The liquid being unlimited, the variation of pressure may be neglected so that the pressure is taken as constant everywhere. The equations of motion then take the forms:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \tag{1}$$

and

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0 \tag{2}$$

where (u, v) are the components of velocity and ν the kinematic viscosity.

Initially at the beginning of the motion, the diffusion far outweighs the convection *i.e.*, the convection terms in the acceleration can be neglected compared with $\partial u/\partial t$. The equation for the first approximation to u is

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2}. \tag{3}$$

We must seek a solution of this equation such that when $t \to 0$, $u \to 0$ everywhere except on the line $y = 0$ where there is some impulsive disturbance and on account of the symmetry $\partial u/\partial y = 0$ on $y = 0$.

The particular solution of (3) satisfying the above conditions is

$$u = \frac{2u_1}{(\pi t)^{\frac{1}{2}}}e^{-\frac{y^2}{4\nu t}} \tag{4}$$

where u_1 can be taken as a function of x or is a constant.

v is to be found from the equation of continuity, namely (2), subject to the condition that $v = 0$ when $y = 0$. From (4) and (2) it is easily found that

$$v = -\frac{4\nu^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}\frac{du_1}{dx}\int_0^\eta e^{-\eta^2}d\eta \tag{5}$$

where

$$\eta = +\frac{y}{2(\nu t)^{\frac{1}{2}}}$$

and u_1 is taken as a function of x, To find a second approximation, let us denote by u^1, v^1 the terms that must be added to the first approximations given by (4) and (5); and to find u^1, it is sufficient to solve the equation

$$\nu\frac{\partial^2 u^1}{\partial y^2}-\frac{\partial u^1}{\partial t}=u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} \tag{6}$$

where u and v have their values as given by (4) and (5) for the first approximation.

The equation (6) easily transforms into

$$\frac{\partial^2 u^1}{\partial \eta^2}+2\eta\frac{\partial u^1}{\partial \eta}=\frac{16}{\pi}u_1\frac{du_1}{dx}\left\{e^{-2\eta^2}+2\eta e^{-\eta^2}\int_0^\eta e^{-\eta^2}d\eta\right\}$$

or

$$\frac{\partial^2 u^1}{\partial \eta^2}+2\eta\frac{\partial u^1}{\partial \eta}=\frac{16}{\pi}u_1\frac{du_1}{dx}\left\{e^{-2\eta^2}+\pi^{\frac{1}{2}}\eta e^{-\eta^2}\operatorname{erf}(\eta)\right\} \tag{7}$$

where

$$\operatorname{erf}(\eta)=\frac{2}{\pi^{\frac{1}{2}}}\int_0^\eta e^{-\eta^2}d\eta. \tag{8}$$

To solve the equation (7), we put

$$u^1=u_1\frac{du_1}{dx}f(\eta)$$

then

$$\frac{d^2f}{d\eta^2}+2\eta\frac{df}{d\eta}=\frac{16}{\pi}e^{-2\eta^2}+\frac{16}{\pi^{\frac{1}{2}}}\eta e^{-\eta^2}\operatorname{erf}(\eta). \tag{9}$$

The solution of this equation is

$$f=2\{\operatorname{erf}(\eta)\}^2-\frac{4}{\pi^{\frac{1}{2}}}\eta e^{-\eta^2}\operatorname{erf}(\eta)-\frac{4}{\pi}e^{-2\eta^2}+2A\operatorname{erf}(\eta)+2B \tag{10}$$

where A and B are constants to be determined from the conditions of the problem.

Thus upto second approximation we have

$$u=\frac{2u_1}{(\pi t)^{\frac{1}{2}}}e^{-\eta^2}+2u_1\frac{du_1}{dx}\left[\{\operatorname{erf}(\eta)\}^2-\frac{2}{\pi^{\frac{1}{2}}}\eta e^{-\eta^2}\operatorname{erf}(\eta)-\frac{2}{\pi}e^{-2\eta^2}+A\operatorname{erf}(\eta)+B\right]. \tag{11}$$

Conditions to be satisfied are

$$u=0 \text{ when } \eta=\infty, \text{ and } \partial u/\partial\eta=0 \text{ when } \eta=0. \tag{12}$$

These give

$$1+A+B=0, \text{ and } A=0$$

so that

$$A=0 \text{ and } B=-1. \tag{13}$$

Hence from (11) with the help of (13), we get

$$u=\frac{2u_1}{(\pi t)^{\frac{1}{2}}}e^{-\eta^2}+2u_1\frac{du_1}{dx}\left[\{\operatorname{erf}(\eta)\}^2-\frac{2}{\pi^{\frac{1}{2}}}\eta e^{-\eta^2}\operatorname{erf}(\eta)-\frac{2}{\pi}e^{-2\eta^2}-1\right]. \tag{14}$$

This will give u when t is not large.

To determine the form of u_1, we note that it must be such a function of x as will make u finite at the origin except when $t = 0$ and zero at infinity. If we take the form $u_1 = x^n$ then we see that for positive values of n, u is negative for small values of x and positive for large values; and for negative values of n, u will be infinite at the origin. Therefore such forms will not be suitable for our purpose. A probable form is $u_1 = e^{-x}$. This will make u finite at the origin when t is not zero and zero at infinity. With this form for u_1, we have

$$u = \frac{2e^{-x}}{(\pi t)^{\frac{1}{2}}} e^{-\eta^2} - 2e^{-2x}\left[\{\operatorname{erf}(\eta)\}^2 - \frac{2}{\pi^{\frac{1}{2}}}\eta e^{-\eta^2} \operatorname{erf}(\eta) - \frac{2}{\pi} e^{-2\eta^2} - 1\right]. \tag{15}$$

This has been evaluated numerically for different values of t and of x and y and the result of this evaluation is given in the following tables.

$t = 0{\cdot}5$

η	$y/\nu^{\frac{1}{2}}$	$(u)_{x=0}$	$(u)_{x=1}$	$(u)_{x=2}$
0	0·0000	4·8690	3·3602	0·2758
1	1·4142	2·0386	0·4123	0·1061
2	2·8284	0·0903	0·0160	0·0051
3	4·2426	0·0011	0·0002	0·0000*

* Practically

$t = 1$

η	$y/\nu^{\frac{1}{2}}$	$(u)_{x=0}$	$(u)_{x=1}$	$(u)_{x=2}$
0	0	4·4016	0·8580	0·2125
1	2	1·8667	0·3491	0·0828
2	4	0·0818	0·0128	0·0039
3	6	0·0011	0·0002	0·0000*

* Practically

From the above tables we see that the thickness of the liquid layer disturbed is little higher for small values of x than for large values. But for all practical purposes

GASES WITH A PRESSURE-DENSITY RELATION—IMPOSSIBILITY OF CONFOCAL DENSITY DISTRIBUTIONS

By

N. L. Ghosh

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Introduction

In two previous papers (Ghosh, 1948, 1949) we proved the impossibility of density-distributions in similar ellipsoids in the case of a mass of gas, where the pressure depends on the density alone, rotating steadily about a fixed axis. In the present paper we propose to prove the same impossibility for density-distributions in confocal ellipsoids. We shall prove the theorem for constant as well as for variable angular velocity and show that the result is true for any gravitational field whatsoever whether due to self-gravitation of the rotating mass alone or the same with external gravitating fields.

The most interesting point in the proof given in this paper as well as in the two previous papers, mentioned, is that no external boundary conditions has explicitly been made use of. The implication is obvious, namely, that ellipsoidal density-stratifications is nowhere possible in a steadily rotating mass of fluid in which the pressure depends entirely on the density—such stratifications being inconsistent with the system of equations involving Poisson's equation and the equations of motion. In all previous proofs of this theorem the potential function for an ellipsoidal body with definite ellipsoidal stratifications has been utilised and the usual boundary-conditions have also been taken care of. The present result is thus a much more general one than what is given by the theorems of Hamy (1889), Véronnet (1912) and Dive (1930).

1

The conditions of steady rotational motion with axial symmetry reduce the dynamical equations to

$$\omega^2 r = -\frac{\partial \Phi}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} \tag{1.1}$$

$$0 = -\frac{\partial \Phi}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z}, \tag{1.2}$$

where the potential Φ is given by Poisson's equation

$$\nabla^2\Phi \equiv \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\Phi = -4\pi G\rho. \tag{1.3}$$

Let us assume generally that

$$\int \frac{dp}{\rho} = f(\rho). \tag{1.4}$$

When 'ω' is constant, we have from (1.1)—(1.4)

$$\nabla^2 f(\rho) + 4\pi G\rho = 2\omega^2. \tag{1.5}$$

When 'ω' is variable, we have from (1.2), (1.3) and (1.4)

$$\nabla^2 \left\{\frac{\partial}{\partial z} f(\rho)\right\} + 4\pi G . \frac{\partial \rho}{\partial z} = 0. \tag{1.6}$$

We shall now show that the assumption of confocal distribution for the density ρ is inconsistent with (1.5) or (1.6) according as ω is assumed respectively to be constant or variable.

For this purpose we shall transform the equations to elliptic co-ordinates in the meridian plane.

2

Let us put

$$\left.\begin{aligned} z &= k\mu\zeta \\ r &= k(1-\mu^2)^{\frac{1}{2}}(1+\zeta^2)^{\frac{1}{2}} \end{aligned}\right\}, \tag{2.1}$$

so that, ζ = constant, are ellipses

$$\frac{r^2}{1+\zeta^2} + \frac{z^2}{\zeta^2} = k^2, \tag{2.2}$$

and μ = constant, are the hyperbolas

$$\frac{r^2}{1-\mu^2} - \frac{z^2}{\mu^2} = k^2 \tag{2.3}$$

both systems being confocal with foci at $(\pm k, 0)$.

In the present case, taking ζ = const, μ = const and ϖ (azimuth) = const, to be a set of orthogonal surfaces we find that the Laplace operater reduces to (Lamb, 1916)

$$\nabla^2 \equiv \frac{1}{k^3(\zeta^2+\mu^2)} \left[\frac{\partial}{\partial\mu}\left\{k(1-\mu^2)\frac{\partial}{\partial\mu}\right\} + \frac{\partial}{\partial\zeta}\left\{k(1+\zeta^2)\frac{\partial}{\partial\zeta}\right\}\right]. \tag{2.4}$$

3

I. *Case of constant angular velocity* (ω = const.). Assuming,

$$\rho = \psi(\zeta) \tag{3.1}$$

and

$$f\{\psi(\zeta)\} = \chi(\zeta) \tag{3.2}$$

we have from (1.5)

$$\nabla^2\chi(\zeta) + 4\pi G . \psi(\zeta) = 2\omega^2. \tag{3.3}$$

Hence, from (2.4)

$$\frac{1}{k^2(\zeta^2+\mu^2)}\frac{d}{d\zeta}\left\{(1+\zeta^2)\frac{d\chi(\zeta)}{d\zeta}\right\} = 2\omega^2 - 4\pi G.\psi(\zeta)$$

or

$$\frac{\dfrac{d}{d\zeta}\left\{(1+\zeta^2)\dfrac{d}{d\zeta}\chi(\zeta)\right\}}{\{2\omega^2 - 4\pi G\psi(\zeta)\}} = k^2(\zeta^2+\mu^2). \tag{3.4}$$

As k is a constant and the left hand side cannot but be a function of ζ alone, equation (3.4), involving μ on the right hand side is obviously impossible unless the density is a constant (Maclaurin's Spheroid).

Thus "for a compressible fluid under the law $p = F(\rho)$, rotating steadily in relative equilibrium about a fixed axis, the distribution of density in confocals given by (2.2) is impossible".

II. *Case of variable angular velocity* (*ω variable*).

Assuming (3.1) and (3.2) to hold in this case also, (1.6) becomes

$$\frac{1}{k^2(\zeta^2+\mu^2)}\left[\frac{\partial}{\partial\mu}\left\{(1-\mu^2)\frac{\partial T}{\partial\mu}\right\}+\frac{\partial}{\partial\zeta}\left\{(1+\zeta^2)\frac{\partial T}{\partial\zeta}\right\}\right]+4\pi G.\psi'(\zeta)\frac{\partial\zeta}{\partial z} = 0 \tag{3.5}$$

where

$$\frac{\partial}{\partial z}\chi(\zeta) = T. \tag{3.6}$$

But, from (2.1), we have

$$\frac{\partial\zeta}{\partial z} = \frac{\mu(1+\zeta^2)}{k(\mu^2+\zeta^2)} \tag{3.7}$$

and hence,

$$T = \frac{\mu}{\mu^2+\zeta^2}.P \tag{3.8}$$

where

$$P = \frac{\chi'(\zeta)(1+\zeta^2)}{k}. \tag{3.9}$$

Substituting in (3.5) and simplifying we have

$$\frac{\mu}{k^2(\mu^2+\zeta^2)}\left[\frac{1+\zeta^2}{\mu^2+\zeta^2}P'' - \frac{2(1-\mu^2)\zeta}{(\mu^2+\zeta^2)^2}P' - 2\,\frac{\{\zeta^4+\zeta^2+\mu^2(1-3\zeta^2)-4\}}{(\mu^2+\zeta^2)^3}P\right]$$

$$+4\pi G.\psi'(\zeta)\,\frac{\mu(1+\zeta^2)}{k(\mu^2+\zeta^2)} = 0 \tag{3.10}$$

where P' and P'' denote the first and second derivatives of P with respect to ζ.

Simplified and arranged in powers of μ, (3.10) reduces to

$$\mu^6X_1+\mu^4X_2+\mu^2X_3+X_4 = 0 \tag{3.11}$$

where the co-efficients X_1, X_2, etc. are given by

$$\left.\begin{aligned}
X_1 &= 4\pi Gk\psi'(\zeta)(1+\zeta^2)\\
X_2 &= (1+\zeta^2)P'' + 2\zeta P' + 3X_1\zeta^2\\
X_3 &= (1+\zeta^2)P''2\zeta^2 + 2\zeta P'(1-\zeta^2) - 2(1-3\zeta^2)P + 3\zeta^4 X_1\\
X_4 &= (1+\zeta^2)P''\zeta^4 - 2\zeta^3 P' - 2\{\zeta^2(1+\zeta^2)-4\}P + X_1\zeta^6
\end{aligned}\right\}. \tag{3.12}$$

For any possible case of the type of motion assumed, equation (3.11) must hold for all values of μ^2; hence all the co-efficients X_1, X_2, X_3, X_4 must vanish simultaneously.

But $X_1 = 0$ gives $\rho =$ constant, which though ensures the vanishing of all the X's, precludes any variation of the density. Hence, in this case also, the distribution of density in confocal ellipsoids is not possible.

In conclusion, I offer my grateful thanks to Prof. N. R. Sen for his unfailing encouragement in the preparation of this work.

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ON THE SCATTERING OF ELECTROMAGNETIC WAVES BY FREE ELECTRON—I. CLASSICAL THEORY

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INTRODUCTION

The main object of this paper is to study rigorously the scattering of electromagnetic waves by unbounded electron, from the standpoint of Electromagnetic Theory of Maxwell-Lorentz. With this object in view, the motion of the electron has been taken to be described by Lorentz equation of motion of electron. The relativistic form of this equation has been solved completely. The scattering is then determined from the well-known Lienard-Wiechert potential corresponding to the velocity of the electron obtained from the solution of the above equation. It is to be noted here that scattered radiation is seen to be not exactly with the same frequency as that of the incident but is changed by a factor which is different from unity by terms consisting of e^2/m^2c^4 and higher powers and of intensity of incident radiation. Further the scattered radiation is composed of a spectrum of frequencies, which are integral multiples of a basic frequency. The ratios of the intensity of scattered radiation to that of incident radiation are in all cases seen to be dependent on the incident intensity. The usual Thomson scattering is only the first order scattering with only first power in e^2/m^2c^4.

The Lorentz-product of two four vectors A and B is denoted by (AB) and defined by

$$(AB) = A_0B_0-(\mathbf{A}.\mathbf{B}) = A_0B_0-A_xB_x-A_yB_y-A_zB_z,$$

where we mean by A_0 and $\mathbf{A}$ the time and space components respectively of the four vector A.

THE MOTION OF THE ELECTRON

We take the incident radiation to be monochromatic of frequency ν and in a direction given by the unit vector $\mathbf{n}$. The incident radiation field is described by the Lorentz four potential

$$A \cos 2\pi\nu\left[t-\frac{(\mathbf{n}.\mathbf{r})}{c}\right] = A \cos \nu\theta$$

where

$$\theta = 2\pi\left[t-\frac{(\mathbf{n}.\mathbf{r})}{c}\right]. \tag{1}$$

The electric vector $\mathbf{E}$ and magnetic vector $\mathbf{H}$ are consequently given by

$$\mathbf{E} = \frac{2\pi\nu}{c}(\mathbf{A} - \mathbf{n}A_0)\sin\nu\theta \tag{2}$$

$$\mathbf{H} = \frac{2\pi\nu}{c}\mathbf{n}\times\mathbf{A}.\sin\nu\theta. \tag{3}$$

In view of the fact that we are to solve the equation of motion which is evidently a three dimensional one, it is convenient to keep the symmetry by introducing a new field vector **a**, defined by

$$\mathbf{a} = \mathbf{A} - \mathbf{n}A_0. \tag{4}$$

This vector **a** on account of Lorentz condition

$$(An) = 0 \tag{5}$$

is normal to the direction of incidence **n**; further this vector is never zero for a non-vanishing field, as

$$a^2 = (\mathbf{a}.\mathbf{a}) = -(AA) \tag{6}$$

is directly proportional to the field intensity.

With this newly defined vector **a**, **E** and **H** can be written as

$$\mathbf{E} = \frac{2\pi\nu}{c}\mathbf{a}\sin\nu\theta, \tag{7}$$

$$\mathbf{H} = \frac{2\pi\nu}{c}\mathbf{n}\times\mathbf{a}.\sin\nu\theta. \tag{8}$$

The relativistic form of the Lorentz equation of motion for the electron

$$\frac{d}{dt}\left[\frac{m\mathbf{U}}{\sqrt{\left[1-\frac{(\mathbf{U}.\mathbf{U})}{c^2}\right]}}\right] = e\left[\mathbf{E} + \frac{1}{c}.\mathbf{U}\times\mathbf{H}\right] \tag{9}$$

where **U** is the velocity of the electron

$$\mathbf{U} = \frac{d\mathbf{r}}{dt}. \tag{10}$$

Substituting the values of **E** and **H** from (7) and (8) in (9)

$$\frac{d}{dt}\left[\frac{m\mathbf{U}}{\sqrt{\left[1-\frac{(\mathbf{U}.\mathbf{U})}{c^2}\right]}}\right] = \frac{2\pi\nu e}{c}\left[\mathbf{a} + \frac{1}{c}.\mathbf{U}\times(\mathbf{n}\times\mathbf{a})\right]\sin\nu\theta. \tag{11}$$

Now introducing the velocity **V** in the unit of velocity of light c

$$c\mathbf{V} = \mathbf{U} \tag{12}$$

and breaking up the vector product in (11) we get

$$\frac{d}{dt}\left[\frac{\mathbf{V}}{\sqrt{(VV)}}\right] = 2\pi\nu\lambda\left[\mathbf{a}(Vn) + \mathbf{n}(\mathbf{V}.\mathbf{a})\right]\sin\nu\theta \tag{13}$$

where for brevity we have put

$$\lambda = \frac{e}{mc^2}. \tag{14}$$

Before proceeding to solve the equation (13), we like to mention that without any loss of generality we can take the initial velocity of the electron to be zero; because if the initial velocity of the electron is different from zero, we can make it zero by effecting a Lorentz transformation, which keeps all our results invariant, since all our calculations are effected in an invariant manner. Further we can choose the origin of the coordinate system such that $\theta = 0$ at $t = 0$. Any other choice would make all our expressions unnecessarily involved. Our aim is to study scattering; as such only oscillatory terms are of interest and any arbitrary fixing of the origin will bring only additive constants in the solution.

Let $\mathbf{j}$ be a unit vector normal to the $(\mathbf{n}\text{-}\mathbf{a})$ plane. On forming scalar product of both sides of the equation (13) with $\mathbf{V}$, $\mathbf{n}$, $\mathbf{a}$ and $\mathbf{j}$ respectively, we get

$$\frac{d}{dt}\left[\frac{1}{\sqrt{(VV)}}\right] = 2\pi\nu\lambda(\mathbf{V}.\mathbf{a}) \sin \nu\theta \tag{15}$$

$$\frac{d}{dt}\left[\frac{(\mathbf{V}.\mathbf{n})}{\sqrt{(VV)}}\right] = 2\pi\nu\lambda(\mathbf{V}.\mathbf{a}) \sin \nu\theta \tag{16}$$

$$\frac{d}{dt}\left[\frac{(\mathbf{V}.\mathbf{a})}{\sqrt{(VV)}}\right] = 2\pi\nu\lambda a^2(Vn) \sin \nu\theta \tag{17}$$

$$\frac{d}{dt}\left[\frac{(\mathbf{V}.\mathbf{j})}{\sqrt{(VV)}}\right] = 0 \tag{18}$$

where use has been made of the relation

$$\left(\mathbf{V}.\frac{d}{dt}\left[\frac{\mathbf{V}}{\sqrt{(VV)}}\right]\right) = \frac{d}{dt}\left[\frac{1}{\sqrt{(VV)}}\right]. \tag{19}$$

The equation (18) points out that the motion is confined in the $(\mathbf{n}\text{-}\mathbf{a})$ plane. From (15) and (16) we get

$$\frac{d}{dt}\left[\frac{1-(\mathbf{V}.\mathbf{n})}{\sqrt{(VV)}}\right] = 0. \tag{20}$$

The integral of (20) furnishes us with a very important relation, which as we see later gives us the main clue for further integration. From (20) we have

$$\frac{(Vn)}{\sqrt{(VV)}} = 1 \tag{21}$$

since the constant of integration is unity. This integral corresponds to the momentum integral of particle dynamics and physically it states that the difference of energy and the component of the momentum of the electron along the direction of incidence is constant and is equal to the rest-energy of the electron if the electron is initially at rest.

We observe from (1)

$$2\pi(Vn) = \frac{d\theta}{dt}. \tag{22}$$

Hence (17) can easily be integrated, giving

$$\frac{(\mathbf{V}.\mathbf{a})}{\sqrt{(VV)}} = \lambda a^2 - \lambda a^2 \cos\nu\theta. \tag{23}$$

Now the original equation (13) may be written with the help of (21), (22) and (23) in the following form

$$\frac{d}{dt}\left[\frac{\mathbf{V}}{\sqrt{(VV)}}\right] = -\lambda\mathbf{a}\frac{d}{dt}(\cos\nu\theta) + \mathbf{n}\lambda 2\pi\nu(Vn)[\lambda a^2 - \lambda a^2\cos\nu\theta]\sin\nu\theta. \tag{24}$$

This equation (24) easily admits of an integral,

$$\frac{\mathbf{V}}{\sqrt{(VV)}} = \lambda\mathbf{a} + \tfrac{3}{4}\lambda^2 a^2\mathbf{n} - \lambda\mathbf{a}\cos\nu\theta - \lambda^2 a^2\mathbf{n}\cos\nu\theta + \tfrac{1}{4}\lambda^2 a^2\mathbf{n}\cos 2\nu\theta. \tag{25}$$

We see from (21) that

$$\frac{1}{\sqrt{(VV)}} = 1 + \frac{(\mathbf{V}.\mathbf{n})}{\sqrt{(VV)}}. \tag{26}$$

Hence by forming scalar product of both sides of (25) with $\mathbf{n}$, we have

$$\frac{1}{\sqrt{(VV)}} = 1 + \tfrac{3}{4}\lambda^2 a^2 - \lambda^2 a^2(\cos\nu\theta - \tfrac{1}{4}\cos 2\nu\theta) \tag{27}$$

Equations (25) and (27) give the velocity of the electron. The expression for velocity may further be integrated yielding

$$\frac{\mathbf{r}}{c} = \frac{1}{2\pi}(\lambda\mathbf{a} + \tfrac{3}{4}\lambda^2 a^2\mathbf{n})\theta - \frac{1}{2\pi\nu}(\lambda\mathbf{a} + \lambda^2 a^2\mathbf{n})\sin\nu\theta + \frac{\lambda^2 a^2}{16\pi\nu}\mathbf{n}\sin 2\nu\theta. \tag{28}$$

This is the integral of the equation of motion, but this expression for $\mathbf{r}$ is in terms of θ which is again a linear function of $\mathbf{r}$. But our object is to determine $\mathbf{r}$ in terms of t, however we can do so by expressing θ in terms of t only. For this purpose, we proceed in the following way.

On forming scalar product of both sides of (28) with $\mathbf{n}$, we get

$$\frac{(\mathbf{n}.\mathbf{r})}{c} = \frac{3}{4}\frac{\lambda^2 a^2}{2\pi}\theta - \frac{\lambda^2 a^2}{2\pi\nu}[\sin\nu\theta - \tfrac{1}{8}\sin 2\nu\theta]. \tag{29}$$

Consequently from (1)

$$2\pi\nu t = (1 + \tfrac{3}{4}\lambda^2 a^2)\nu\theta - \lambda^2 a^2(\sin\nu\theta - \tfrac{1}{8}\sin 2\nu\theta). \tag{30}$$

For shortness, we put

$$\nu' = \nu(1 + \tfrac{3}{4}\lambda^2 a^2)^{-1}, \tag{31}$$

$$z = \lambda^2 a^2(1 + \tfrac{3}{4}\lambda^2 a^2)^{-1} \tag{32}$$

so that equation (30) takes the form

$$2\pi\nu' t = \nu\theta - z(\sin\nu\theta - \tfrac{3}{8}\sin 2\nu\theta). \tag{33}$$

Now to express $\nu\theta$ in terms of t is the problem of inversion of the series. The form of equation (33) reminds one, of the well-known Kepler equation between eccentric anomaly and mean anomaly of planetary motion in celestial dynamics. Euation (17) however contains an additional term in $\sin 2\nu\theta$, than that of Kepler. The inversion of (33) and that of $\cos p\nu\theta$, $\sin p\nu\theta$ has been worked out in Appendix I. Here we give the results:

$$\nu\theta = 2\pi\nu' t + 2\sum_{1}^{\infty}\chi_n(nz)\frac{\sin 2\pi\nu' nt}{n} \tag{34}$$

$$\sin p\nu\theta = p\sum_{1}^{\infty}[\chi_{n-p}(nz) + \chi_{n+p}(nz)]\frac{\sin 2\pi n\nu' t}{n} \tag{35}$$

$$\cos p\nu\theta = p\sum_{1}^{\infty}[\chi_{n-p}(nz) - \chi_{n+p}(nz)]\frac{\cos 2\pi n\nu' t}{n} \tag{36}$$

where $\chi_n(x)$ is defined by

$$\chi_n(x) = \frac{1}{2\pi}\int_0^{2\pi}\exp\{-in\varphi + ix(\sin\varphi - \tfrac{3}{8}\sin\varphi)\}d\varphi \tag{37}$$

and χ_n's satisfy the recurrence formula

$$\frac{2n}{x}\chi_n(x) = [\chi_{n+1}(x) + \chi_{n-1}(x)] - \tfrac{3}{4}[\chi_{n+2}(x) + \chi_{n-2}(x)]. \tag{38}$$

Substituting the values of $\nu\theta$, $\sin\nu\theta$ and $\sin 2\nu\theta$ from (34) and (35) in the expression for $\mathbf{r}$ given by equation (28) and on utilising the recurrence relation (38), we have

$$\begin{aligned} 2\pi\nu\frac{\mathbf{r}}{c} = {} & (\lambda\mathbf{a} + \tfrac{3}{4}\lambda^2 a^2\mathbf{n})2\pi\nu' t \\ & + \sum_{1}^{\infty}\left[\lambda\mathbf{a}\{2\chi_n(nz) - \chi_{n+1}(nz) - \chi_{n-1}(nz)\} + \lambda^2 a^2\mathbf{n}\left(\frac{3}{2} - \frac{2n}{z}\right)\chi_n(nz)\right]\frac{\sin 2\pi n\nu' t}{n}; \end{aligned} \tag{39}$$

This equation completely determines the position of the electron at any instant. It points out that the oscillatory part of the motion is described by frequencies which are integral multiples of ν'. This fundamental frequency differs from the incident frequency by a factor different from unity. It ought to be mentioned here that this factor namely $(1 + \tfrac{3}{4}\lambda^2 a^2)^{-1}$ depends on the incident intensity and frequency and differ from unity by insignificantly small quantity. Now differentiating (39) we get the velocity of the electron directly in terms of t,

$$\mathbf{V} = \sum_{-\infty}^{\infty}\mathbf{V}_n e^{i2\pi n\nu' t} \tag{40}$$

where

$$\mathbf{V}_n = \frac{\nu'}{2\nu}\left[\lambda\mathbf{a}\{2\chi_n(nz) - \chi_{n+1}(z) + \chi_{n-1}(z)\} + \lambda^2 a^2\mathbf{n}\left(\frac{3}{2} - \frac{2n}{z}\right)\chi_n(nz)\right], \qquad 4) \qquad 1)$$

$$\mathbf{V}_n = \mathbf{V}_{-n} \quad (n \text{ is positive}) \tag{41'}$$

and

$$\mathbf{V}_0 = \frac{\nu'}{\nu}(\lambda \mathbf{a} + \tfrac{3}{4}\lambda^2 a^2 \mathbf{n}). \tag{42}$$

Before concluding this section, it is important to mention that the process of solution of the equation of motion shows that we need not restrict the incident radiation to be monochromatic, but it may be easily extended to the case, where the incident radiation consists of more than one frequency but all of them in the same direction, *i.e.*, when the field potential may be represented by

$$\sum_s A_s \cos \nu'_s \theta. \tag{43}$$

The main point to be noted here is that the incident field may be determined by a single variable θ.

THE RADIATION POTENTIAL AND SCATTERING

The radiation potential due to an electron moving with a velocity $\mathbf{U}$ is, according to Lienard and Wiechert, given by

$$A_0' = \frac{e}{r}\left[\left(1 + \frac{(\mathbf{U}.\mathbf{r})}{rc}\right)^{-1}\right]_{t-r/c} \tag{43}$$

$$\mathbf{A}' = \frac{e}{r}\left[\frac{\mathbf{U}}{c}\left(1 + \frac{(\mathbf{U}.\mathbf{r})}{rc}\right)^{-1}\right]_{t-r/c} \tag{44}$$

Let the direction-cosines of the direction of observation be denoted by $\mathbf{n}'$, *i.e.*, $\mathbf{n}' = \mathbf{r}/r$; so that

$$A_0' = \frac{e}{r}\left[[1 + (\mathbf{V}.\mathbf{n}')]^{-1}\right]_{t-r/c} \tag{45}$$

$$\mathbf{A}' = \frac{e}{r}\left[\frac{\mathbf{U}}{c}[1 + (\mathbf{V}.\mathbf{n}')]^{-1}\right]_{t-r/c}. \tag{46}$$

Since $|(\mathbf{V}\mathbf{n}')|$ is always less than unity, we can expand the right-hand sides of (45) and (46) and write the expression in the following form

$$A_0' = \frac{e}{r}[1 - (\mathbf{V}.\mathbf{n}') + (\mathbf{V}.\mathbf{n}')^2 - (\mathbf{V}.\mathbf{n}')^3 + \cdots]_{t-r/c} \tag{47}$$

$$\mathbf{A}' = \frac{e}{r}[\mathbf{V} - \mathbf{V}(\mathbf{V}.\mathbf{n}') + \mathbf{V}(\mathbf{V}\mathbf{n}')^2 \cdots]_{t-r/c}. \tag{47}$$

On substituting the value of $\mathbf{V}$ from (40) and replacing t by $t-r/c$; we get

$$A_0' = \frac{e}{r}\left[1 + \tfrac{1}{2}\sum_{-\infty}^{\infty} A'_{0n} \exp\left\{i2\pi n\nu'\left(t - \frac{(\mathbf{n}'.\mathbf{r})}{c}\right)\right\}\right] \tag{41}$$

$$\mathbf{A}' = \frac{e}{r}\cdot\frac{1}{2}\sum_{-\infty}^{+\infty}\mathbf{A}'_n \exp\left\{i2\pi n\nu'\left(t-\frac{(\mathbf{n}'.\mathbf{r})}{c}\right)\right\} \tag{50}$$

where $\frac{1}{2}A'_{0n}$ and $\frac{1}{2}\mathbf{A}'_n$ stands for

$$\tfrac{1}{2}.A'_{0n} = -(\mathbf{V}_n.\mathbf{n}') + \sum_{-\infty}^{\infty}(\mathbf{V}_k.\mathbf{n}')(\mathbf{V}_{n-k}.\mathbf{n}') - \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}(\mathbf{V}_k.\mathbf{n}')(\mathbf{V}_l.\mathbf{n}')(\mathbf{V}_{n-k-l}.\mathbf{n}') + \cdots \tag{51}$$

$$\tfrac{1}{2}.\mathbf{A}'_n = \mathbf{V}_n - \sum_{-\infty}^{\infty}\mathbf{V}_k(\mathbf{V}_{n-k}.\mathbf{n}') + \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}\mathbf{V}_k(\mathbf{V}_l.\mathbf{n}')(\mathbf{V}_{n-k-l}.\mathbf{n}') - \cdots \tag{52}$$

The expression for radiation potential shows that scattered radiation in any direction is composed of a spectrum of frequency ν', $2\nu'$, $3\nu'$, . . . forming a sequence of harmonics of the fundamental $\nu' = \nu(1+\frac{3}{4}\lambda^2a^2)^{-1}$. Although this frequency is not exactly equal to that of incident radiation but is independent of direction of observation.

The scattered radiation may be classified in the following way. The constant non-oscillatory part is related to the static potential of the electron.

$n = 1$: *First Order Scattering.* The scattered frequency ν' is but for a negligible term is same as the incident frequency.

$n = 2$: *Second Order Scattering* The scattered frequency is $2\nu'$, *i.e.*, double that of first order; and similarly for higher values of n.

Now, let I stand for the intensity of incident radiation and is thus given by

$$I = \frac{\pi\nu^2}{c}a^2 = -\frac{\pi\nu^2}{c}(AA) \tag{53}$$

Again, we observe from (51), (52) and (41) that

$$A'_{0n} = A'_{0,-n}, \quad \mathbf{A}'_n = \mathbf{A}_{-n}$$

so that the expression for radiation potential may be written as

$$A'_0 = \frac{e}{r}\left[1+\tfrac{1}{2}A'_{00}+\sum_{1}^{\infty}A'_{0n}\cos 2\pi\nu\left(t-\frac{(\mathbf{n}'.\mathbf{r})}{c}\right)\right] \tag{54}$$

$$\mathbf{A}'_n = \frac{e}{r}\left[\tfrac{1}{2}\mathbf{A}_0+\sum_{1}^{\infty}\mathbf{A}'_n\cos 2\pi\nu\left(t-\frac{(\mathbf{n}'.\mathbf{r})}{c}\right)\right]. \tag{55}$$

Since there is no phase correlation between the different scattered radiation, the intensity for any particular order of scattering is determined by the corresponding coefficient in the expansion of radiation potential. If I_n is the intensity of nth order scattering we get

$$\frac{I_n}{I} = \frac{e^2n^2\nu'^2}{r^2\nu^2}\frac{(A'_nA'_n)}{(AA)} \tag{56}$$

so that the ratio of I_n/I can be calculated from the knowledge of $(A'_nA'_n)$.

Before going to calculate the intensity ratio for different order we give below the values of $\mathbf{V}_n$ consisting of terms up to second power e/mc^2, *i.e.*, up to λ^2.

$$V_0 = \lambda\mathbf{a}+\tfrac{3}{4}\lambda^2a^2\mathbf{n},$$

$$V_1 = -\tfrac{1}{2}(\lambda\mathbf{a}+\lambda^2a^2\mathbf{n}),$$

$$V_2 = \tfrac{1}{4}\lambda^2a^2\mathbf{n}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

(*a*) FIRST ORDER SCATTERING (THOMSON SCATTERING)

The frequency of scattered radiation is

$$\nu' = \nu(1+\tfrac{3}{4}\lambda^2a^2)^{-1} \tag{21}$$

The scattered frequency is always less than the incident one. When represented in terms of incident intensity it takes the form

$$\nu' = \nu-\frac{3}{4}\frac{e^2}{m^2c^4}\left(\frac{Ic}{\pi\nu}\right). \tag{57}$$

From (51), (52) we get for $n = 1$,

$$A'_{01} = \lambda(\mathbf{a}.\mathbf{n}')+\lambda^2a^2(\mathbf{n}.\mathbf{n}')-2\lambda^2(\mathbf{a}.\mathbf{n}')^2$$

$$\mathbf{A}_1 = -\lambda\mathbf{a}-\lambda^2a^2\mathbf{n}+2\lambda^2a^2(\mathbf{a}.\mathbf{n}').$$

Let φ be the angle between $\mathbf{n}$ and $\mathbf{n}'$, *i.e.*, the angle between the direction of observation and that of incidence, and ψ between the directions $\mathbf{a}$ and $\mathbf{n}'$, so that

$$(A'_1A'_1) = -\lambda^2a^2\sin^2\psi-2\lambda^3a^3(2\cos^3\psi-2\cos\psi+\cos\varphi\cos\psi).$$

Hence we get finally after averaging over both the directions of polarisation of the incident wave

$$\frac{I_1}{I} = \frac{e^4}{m^2c^4}.\frac{1+\cos^2\varphi}{2r^2}\left[1+\frac{e}{mc^2}\sqrt{\left(\frac{Ic}{\pi}\right)}.\frac{F(\varphi)}{2\nu}\right] \tag{58}$$

where

$$F(\varphi) = -\frac{\sin\varphi+\sin 2\varphi+\sin 3\varphi}{1+\cos^2\varphi}. \tag{59}$$

The first factor is the usual Thomson scattering. It is multiplied by a factor differing from unity by a term of the order of e/mc^2. Since $e/mc^2 \ll 1$, the effective contribution due to this term for all available range of frequency and intensity are small. Further it is to be noted here that this part of the distribution function depends on the absolute value of intensity and frequency of incident radiation.

(*b*) SECOND ORDER SCATTERING

The frequency of scattered radiation is $2\nu'$. On retaining, only the first significant power in ea/mc^2, we get, from (51), (52) for $n = 2$

$$A'_{02} = -\tfrac{1}{2}\lambda^2 a^2(\mathbf{n}.\mathbf{n}') + \lambda^2(\mathbf{a}.\mathbf{n}')^2,$$

$$\mathbf{A}_2 = \tfrac{1}{2}\lambda^2 a^2\mathbf{n} - \tfrac{1}{2}\lambda^2\mathbf{a}(\mathbf{a}.\mathbf{n}').$$

Hence

$$(A'_2A'_2) = \tfrac{1}{16}\lambda^4 a^4 \sin^2\varphi(2\cos\varphi - \sin^2\varphi - 3)$$

when averaged over both the directions of polarisations of incident waves. Hence we get finally

$$\frac{I_2}{I} = \frac{1}{r^2}\frac{e^6}{m^4c^8}\left(\frac{Ic}{\pi\nu^2}\right)F'(\varphi) \tag{60}$$

where

$$F'(\varphi) = \tfrac{1}{8}\sin^2\varphi(3 + \sin^2\varphi - 2\cos\varphi) \tag{61}$$

which gives the intensity of second order scattering. It shows that ratio of the intensities is of the order of e^6/m^4c^8 and depends also on the intensity and frequency of incident wave.

CONCLUSION

The results which we derive here, differ only to a very small extent from the previous results and hence may not be of much importance from physical standpoint, but we will require these results to compare with them, the corresponding formula obtained from modern quantum mechanics, which will form the subject of the second part of this series.

However, an important point remains to be mentioned here. The scattering truly depends on two parameters e/mc^2 and e; and not only on e^2/m^2c^4, which is the case if we restrict ourselves only to the first term of first order scattering. The appearance of these two different parameters is due to the fact that scattering of electromagnetic waves by electrons may be looked upon as two distinct processes. Firstly the electron moves in the field obeying the equation of motion and secondly by virtue of this motion the electron radiates. The presence of the first parameter e/mc^2 is due to the former process and this is really the interaction parameter between the field and electron as can be seen from the equation of motion (13). If we represent the field potential in units of e/mc^2, then equation (13) is independent of any physical constant. Further, the latter process, *i.e.*, the process of radiation depends on the charge of the electron. Consequently scattering process depends on two distinct parameters e/mc^2 and e,

APPENDIX I

Let us consider the relation

$$\tau = \varphi - s(\sin\varphi - \tfrac{1}{8}\sin 2\varphi) \tag{A1}$$

Writing

$$\varphi - \tau = \omega$$

we get

$$\omega = s[\sin(\omega+\tau) - \tfrac{1}{8}\sin 2(\omega+\tau)] \tag{A2}$$

Hence following Lagrange, we can develop ω in a power series of s

$$\varphi = \tau + z(\sin\tau - \tfrac{1}{8}\sin\tau) + \frac{z^2}{2!}\frac{d}{d\tau}(\sin\tau - \tfrac{1}{8}\sin 2\tau)^2 + \cdots + \frac{z^n}{n!}\frac{d^{n-1}}{d\tau^{n-1}}(\sin\tau - \tfrac{1}{8}\sin 2\tau)^n + \cdots \quad \text{(A3)}$$

This series converges so long as the ratio

$$\left| \frac{z(\sin\tau - \frac{1}{8}\sin 2\tau)}{\varphi - \tau} \right|$$

is less than unity. On putting

$$\varphi - \tau = re^{i\theta}.$$

The condition of convergency takes the form

$$\frac{|z|^2}{r^2}[\{\sin(\tau + re^{i\theta}) - \tfrac{1}{8}\sin 2(\tau + re^{i\theta})\}\{\sin(\tau + re^{-i\theta}) - \tfrac{1}{8}\sin 2(\tau + re^{-i\theta})\}] < 1.$$

It can be easily shown that

$$[\{\sin(\tau + re^{i\theta}) - \tfrac{1}{8}\sin 2(\tau + re^{i\theta})\}\{\sin(\tau + re^{-i\theta}) - \tfrac{1}{8}\sin 2(\tau + re^{-i\theta})\}]$$
$$\leq \cosh^2 r + \tfrac{1}{64}\cosh^2 2r + \tfrac{1}{4}\cosh^2 \tfrac{1}{2}r.$$

Hence the ratio is less than unity, so long as

$$\frac{|z|}{r}[\cosh^2 r + \tfrac{1}{64}\cosh^2 2r + \tfrac{1}{4}\cosh^2 \tfrac{1}{2}r]^{\frac{1}{2}} < 1.$$

The series converges consequently for

$$|z| < \frac{r}{[\cosh^2 r + \frac{1}{64}\cosh^2 2r + \frac{1}{4}\cosh^2 \frac{1}{2}r]^{\frac{1}{2}}} = f(r). \quad \text{(A4)}$$

The maximum value of $f(r)$ is thus the limit of $|z|$. The value of r for which $f(r)$ attains the maximum value is determined by solving numerically the equation

$$\frac{df(r)}{dr} = 0$$

which amounts to find out a root of the numerical equation

$$\tfrac{81}{64} + \cosh 2r + \tfrac{1}{64}\cosh 4r + \tfrac{1}{4}\cosh r = r(\sinh 2r + \tfrac{1}{32}\sinh 4r + \tfrac{1}{8}\sinh r). \quad \text{(A5)}$$

The root of this equation is

$$r = 1{\cdot}1874$$

to which correspond the maximum value of $f(r)$

$$f(r) = 0.575.$$

Thus the inverse series (A3) converges so long as

$$|\lambda^2(AA)[1 - \tfrac{3}{4}\lambda^2(AA)]^{-1}| < 0.575$$

that is,

$$\left| \frac{e^2}{mc^2}(AA) \right| < 0.402 \quad \text{(A6)}$$

which is well within all the available range of intensity and frequency.

Since we are interested in the scattering problem we better seek for a Fourier series in $2\pi\nu' t$ for θ. We observe that such a series is nothing but the series (A8) whose terms are to be rearranged. We are permitted to do so within the limit (A6) or in other words we assert the existence of such an expasion in this region.

In order to find the required Fourier series we proceed in a rather convenient way. Firstly consider the expansion

$$\exp\{iz(\sin\varphi-\tfrac{1}{8}\sin 2\varphi)\} = \sum_{-\infty}^{\infty}\chi_n(z)\exp(in\varphi) \tag{A7}$$

so that

$$\chi_n(z) = \frac{1}{2\pi}\int_0^{2\pi}\exp\{-in\varphi+iz(\sin\varphi-\tfrac{1}{8}\sin 2\varphi)\}d\varphi. \tag{A8}$$

From (A7) we note that

$$\chi_n(z) = \sum_{-\infty}^{\infty}J_{n-2q}(z)J_q(-z/8) = \sum_{-\infty}^{\infty}J_{n+2q}(z)J_q(z/8) \tag{A9}$$

and further

$$\chi_{-n}(-z) = \chi_n(z) \tag{A10}$$

where J_p's are Bessel functions of pth order. From (A1)

$$\frac{d\varphi}{d\tau} = \frac{1}{1-z(\cos\varphi-\frac{1}{4}\cos 2\varphi)} = \sum_{-\infty}^{\infty}\psi_n(z)e^{in\tau} \tag{A11}$$

so that

$$\psi_n = \frac{1}{2\pi}\int_0^{2\pi}e^{-in\tau}\,d\varphi$$

that is

$$\psi_n = \chi_n(nz)$$

from the definition of χ's by (A8).

Now integrating the equation

$$\frac{d\varphi}{d\tau} = \sum_{-\infty}^{\infty}\chi_n(nz)e^{in\tau}$$

we get

$$\varphi = \tau + 2\sum_{1}^{\infty}\chi_n(nz)\,\frac{\sin n\tau}{n}, \tag{A12}$$

from which follows the equation (34) of the text.

Again, let us consider

$$e^{ip\varphi}\frac{d\varphi}{d\tau} = \frac{e^{ip\varphi}}{1-z(\cos\varphi-\frac{1}{4}\cos 2\varphi)} = \sum_{-\infty}^{\infty}\Psi_n(nz)e^{inz} \tag{A13}$$

where $p \neq 0$; we have

$$\Psi_n = \frac{1}{2\pi}\int_0^{2\pi}e^{ip\varphi-in\tau}d\varphi$$

that is,

$$\Psi_n = \chi_{n-p}(nz). \tag{A14}$$

Again integrating the equation

$$e^{ip\varphi}\frac{d\varphi}{d\tau} = \sum_{-\infty}^{\infty}\chi_{n-p}(nz)e^{in\tau}$$

we get

$$e^{ip\varphi} = p\sum_{-\infty}^{\infty}{}' \chi_{n-p}(nz)\frac{e^{in\tau}}{n}, \tag{A15}$$

where the dash in the summation sign means, $n = 0$ is excluded. Finally we get from (A15)

$$\sin p\varphi = p\sum_{-\infty}^{\infty}[\chi_{n-p}(nz)+\chi_{n+p}(nz)]\frac{\sin n\tau}{n} \tag{A16}$$

$$\cos p\varphi = p\sum_{-\infty}^{\infty}[\chi_{n-p}(nz)-\chi_{n+p}(nz)]\frac{\cos n\tau}{n} \tag{A17}$$

from which follows the equations (35) and (36) of the text.

To obtain the recurrence formula between the $\chi_n(z)$ functions of different order we write (A7) in the following form

$$\exp\left\{\frac{z}{2}\left[\left(u-\frac{1}{u}\right)-\frac{1}{8}\left(u^2-\frac{1}{u^2}\right)\right]\right\} = \sum_{-\infty}^{\infty}\chi_n(z)u^n. \tag{A18}$$

Operating $u\frac{\partial}{\partial u}$ on both sides of the equation

$$\left[\frac{z}{2}\left(u+\frac{1}{u}\right)-\frac{z}{4}\left(u^2+\frac{1}{u^2}\right)\right]\sum_{-\infty}^{\infty}\chi_n(z)u^n = \sum_{-\infty}^{\infty}n\,\chi_n(z)u^n. \tag{A19}$$

Now equating the coefficients of u^n of both sides of the equation (A16).

$$\tfrac{1}{2}z[\{\chi_{n+1}(z)+\chi_{n-1}(z)\}-\tfrac{1}{4}\{\chi_{n+2}(z)+\chi_{n+2}(z)\}] = n\chi_n(z) \tag{A20}$$

which is the desired recurrence formula (38) of the text.

The author takes this opportunity to express his gratitude and thanks to Prof. S. N. Bose, for his kind encouragement during the progress of this work.

Note added in the proof on November 21, 1949: A somewhat similar solution of particle equation has been obtained by Taub, A. H. (*Phys. Rev.*, **73**, 786, 1948) in a paper entitled "Orbits of charged particles in constant field," in terms of Lorentz matrices determined by the antisymmetric tensor describing the external field. The author regrets to mention that this paper escaped his previous notice.

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THERMAL STRESSES IN ISOTROPIC CIRCULAR DISKS OF VARYING THICKNESS ROTATING ABOUT A CENTRAL AXIS

By

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Introduction

Thermal stresses are of great practical importance especially in the case of large cylinders such as steam turbine rotors, heavy shafts or large turbine disks (Timoshenko, 1931). The problem of determining thermal stresses in cylinders has been discussed by H. C. Lees (1922), A. Stodola (1924), L. H. Baker (1927), Timoshenko (1931) and others. Thermal stresses in uniform disks have also been determined by several authors (Timoshenko, 1934). A general semigraphical method of determining thermal stresses in rotating disks of variable thickness has recently been found out by Leopold (1948). According to the author the actual profile is to be replaced by a system of disks of uniform thickness. It is further assumed that the radial temperature gradient may be expressed as kr^n where both k and n may have any value and sign, as in the opinion of the author through this type of function it is possible to approximate closely most common temperature gradient curves. In the present paper thermal stresses in isotropic circular disks of varying thickness rotating about a central axis are found out by analytical method, the thickness $2h$ at a distance r from the axis being given by

$$\text{(i)}\quad h = h_0 r^{-\lambda},$$

$$\text{(ii)}\quad h = h_0 \exp(-\lambda r^q).$$

in which h_0, λ are constants and q another constant whose value is given later on.

For disks of profile (i) it is assumed that the temperature gradient may be expressed approximately as kr^n where both k and n may have any value and sign. For the disks of profile (ii) the distribution of temperature throughout the disks is assumed to follow the steady logarithmic law. The material in either case, is assumed to expand according to the linear law. The distribution of temperature is assumed to be constant across the axial thickness of the disk. Further, all variables are assumed symmetrical about the axis of the disk and the disk is considered thin in that all axial stresses are neglected.

1. Statement of the problem

Let z-axis be the axis of the disk, u the mean radial displacement across the thickness of the plate at a distance r from the centre due to stressing and heating, α the coefficient of linear expansion, T the increase in temperature which is a function of the radial distance r and e_r, e_θ respectively the mean radial and tangential strains.

Then, the unit elongations in the radial and tangential directions are

$$e_r = \frac{du}{dr}, \quad e_\theta = \frac{u}{r}. \tag{1.1}$$

Let P and Q denote the mean radial and tangential stresses at a distance r from the centre, $2h$ the thickness at the point, ω the angular velocity and ρ the mass per unit volume of the disk. Then, the equation of motion is (Prescott, 1946).

$$\frac{d}{dr}(rhP) - hQ + \rho h r^2 \omega^2 = 0. \tag{1.2}$$

Observing that the strains e_r, e_θ are due not only to stresses but also to the change in temperature, the mean stress strain relations by Hooke's Law become

$$\begin{aligned} e_r &= \frac{1}{E}(P - \nu Q) + \alpha T \\ e_\theta &= \frac{1}{E}(Q - \nu P) + \alpha T \end{aligned} \tag{1.3}$$

where E = Young's modulus and ν = Poisson's ratio.

The condition of compatibility in the case of deformation symmetrical about the central axis is

$$\frac{d}{dr}(re_\theta) - e_r = 0. \tag{1.4}$$

Substituting in this the values of e_r, e_θ from equations (1.3) and using equation (1.2), we find that P must satisfy the equation

$$r^2 \frac{d^2P}{dr^2} + \left[3 + \frac{r}{h}\frac{dh}{dr}\right] r \frac{dP}{dr} + \left[(2+\nu)\frac{r}{h}\frac{dh}{dr} + r^2 \frac{d}{dr}\left(\frac{1}{h}\frac{dh}{dr}\right)\right] P$$

$$= -(3+\nu)\rho r^2 \omega^2 - E\alpha r \frac{dT}{dr}. \tag{1.5}$$

Also, using equations (1.1) we have, from equations (1.3)

$$\begin{aligned} P &= \frac{E}{1-\nu^2}\left[\frac{du}{dr} + \nu \frac{u}{r}\right] - \frac{E\alpha T}{1-\nu}, \\ Q &= \frac{E}{1-\nu^2}\left[\nu \frac{du}{dr} + \frac{u}{r}\right] - \frac{E\alpha T}{1-\nu}. \end{aligned} \tag{1.6}$$

Substitution of these values in (1.2) gives

$$\frac{d^2u}{dr^2} + \frac{1}{r}\left(1 + \frac{r}{h}\frac{dh}{dr}\right)\frac{du}{dr} - \frac{1}{r^2}\left(1 - \frac{\nu r}{h}\frac{dh}{dr}\right)u = (1+\nu)\alpha\frac{dT}{dr} + \frac{\alpha(1+\nu)}{h}\frac{dh}{dr}T - \frac{(1-\nu^2)}{E}\rho r \omega^2. \tag{1.7}$$

For a uniform disk, if the temperature T is expressed as $T_r = t_0 + kr^n$; where T_r is the temperature at any point distant r from the centre, and t_0 the initial temperature at the absolute centre of the disk, the equation (1.7) reduces to

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}-\frac{1}{r^2}u-(1+\nu)n\alpha kr^{n-1}+\frac{\rho(1-\nu^2)}{E}r\omega^2 = 0, \tag{1.8}$$

which coincides with equation (4) of Leopold's paper (1948).

2. Thermal stresses in rotating disks with thickness varying as $r^{-\lambda}$.

Let

$$h = h_0r^{-\lambda} \tag{2.1}$$

and

$$T = t_0+kr^n \tag{2.2}$$

where T = temperature at a point distant r from the centre of the disk and t_0 = the initial temperature at the absolute centre of the disk.

On substitution of these values of h and T, the equation (1.5) reduces to

$$r^2\frac{d^2P}{dr^2}+(3-\lambda)r\frac{dP}{dr}-(1+\nu)\lambda P = -(3+\nu)\rho r^2\omega^2-E\alpha knr^n. \tag{2.3}$$

The general solution of equation (2.3) may be obtained by adding any particular solution of it to the solution of the corresponding homogeneous equation. From this the general solution of equation (2.3) is found to be

$$P = C_1r^{m_1}+C_2r^{m_2}+Ar^2+Br^n \tag{2.4}$$

where

$$m_1 = \frac{-(2-\lambda)+[(2-\lambda)^2+4\lambda(1+\nu)]^{\frac{1}{2}}}{2},\quad m_2 = \frac{-(2-\lambda)-[(2-\lambda)^2+4\lambda(1+\nu)]^{\frac{1}{2}}}{2}, \tag{2.5}$$

$$A = -\frac{(3+\nu)\rho\omega^2}{8-\lambda(3+\nu)},\quad B = \frac{-E\alpha kn}{n(n+2)-\lambda(n+1+\nu)}. \tag{2.6}$$

and C_1, C_2 are constants of integration to be determined from the boundary conditions. From equation (1.2),

$$Q = \frac{1}{h}\frac{d}{dr}(rhP)+\rho r^2\omega^2.$$

Substituting in this the values of h and P from equations (2.1) and (2.4), we get

$$Q = (1+m_1-\lambda)C_1r^{m_1}+(1+m_2-\lambda)C_2r^{m_2}+(3-\lambda)Ar^2+(1+n-\lambda)Br^n+\rho r^2\omega^2. \tag{2.7}$$

For a disk of radius b with a hole of radius a at the centre and no external forces acting on the edges, the boundary conditions are

$$(P)_{r=a} = 0,\quad (P)_{r=b} = 0.$$

From the equation (2.4), we find that these conditions are satisfied if

$$C_1 = \frac{A(a^{m_2}b^2-a^2b^{m_2})+B(a^{m_2}b^n-a^nb^{m_2})}{a^{m_1}b^{m_2}-a^{m_2}b^{m_1}},$$

$$C_2 = \frac{A(a^2b^{m_1}-a^{m_1}b^2)+B(a^nb^{m_1}-a^{m_1}b^n)}{a^{m_1}b^{m_2}-a^{m_2}b^{m_1}}.$$

Substituting these values of C_1, C_2 in equations (2.4), (2.7) and using equation (2.6), we get

$$P = \frac{(3+\nu)\rho\omega^2}{8-\lambda(3+\nu)}\left[\frac{a^{m_2}b^2-a^2b^{m_2}}{a^{m_2}b^{m_1}-a^{m_1}b^{m_2}}\,r^{m_1}-\frac{a^{m_1}b^2-a^2b^{m_1}}{a^{m_2}b^{m_1}-a^{m_1}b^{m_2}}\,r^{m_2}-r^2\right]$$
$$+\frac{E\alpha kn}{n(n+2)-\lambda(1+n+\nu)}\left[\frac{a^{m_2}b^n-a^nb^{m_2}}{a^{m_2}b^{m_1}-a^{m_1}b^{m_2}}\,r^{m_1}-\frac{a^{m_1}b^n-a^nb^{m_1}}{a^{m_2}b^{m_1}-a^{m_1}b^{m_2}}\,r^{m_2}-r^n\right] \quad (2.8)$$

$$Q = \frac{(3+\nu)\rho\omega^2}{8-\lambda(3+\nu)}\left[(1+m_1-\lambda)\,\frac{a^{m_2}b^2-a^2b^{m_2}}{a^{m_2}b^{m_1}-a^{m_1}b^{m_2}}\,r^{m_1}\right.$$
$$\left.-(1+m_2-\lambda)\,\frac{a^{m_1}b^2-a^2b^{m_1}}{a^{m_2}b^{m_1}-a^{m_1}b^{m_2}}\,r^{m_2}-\frac{1+3\nu}{3+\nu}\,r^2\right]$$
$$+\frac{E\alpha kn}{n(n+2)-\lambda(1+n+\nu)}\left[(1+m_1-\lambda)\,\frac{a^{m_2}b^n-a^nb^{m_2}}{a^{m_2}b^{m_1}-a^{m_1}b^{m_2}}\,r^{m_1}\right.$$
$$\left.-(1+m_2-\lambda)\,\frac{a^{m_1}b^n-a^nb^{m_1}}{a^{m_2}b^{m_1}-a^{m_1}b^{m_2}}\,r^{m_2}-(1+n-\lambda)r^n\right] \quad (2.9)$$

For a uniform disk of radius b with a central hole of radius a, $\lambda = 0$.

Putting $\lambda = 0$ in the equations (2.8) and (2.9), and also noting that when $\lambda = 0$, $m_1 = 0$, $m_2 = -2$, we find that the values of the radial and tangential stresses P and Q for a uniform rotating disk are given by

$$P = \frac{(3+\nu)\rho\omega^2b^2}{8}\left[\frac{a^2}{b^2}+1-\frac{r^2}{b^2}-\frac{a^2}{r^2}\right]$$
$$+\frac{E\alpha k(b^n-a^n)}{(n+2)(b^2-a^2)}\left[\frac{b^{n+2}-a^{n+2}}{b^n-a^n}-\frac{b^2-a^2}{b^n-a^n}\,r^n-\frac{a^2b^2}{r^2}\right], \quad (2.10)$$

$$Q = \frac{(3+\nu)\rho\omega^2b^2}{8}\left[\frac{a^2}{b^2}+1-\frac{1+3\nu}{3+\nu}\,\frac{r^2}{b^2}+\frac{a^2}{r^2}\right]$$
$$+\frac{E\alpha k(b^n-a^n)}{(n+2)(b^2-a^2)}\left[\frac{b^{n+2}-a^{n+2}}{b^n-a^n}-\frac{(1+n)(b^2-a^2)}{b^n-a^n}\,r^n+\frac{a^2b^2}{r^2}\right]. \quad (2.11)$$

which coincide with the results obtained in equations (7*a*) and (7*b*) of Leopold's paper (1948).

3. THERMAL STRESSES IN ROTATING DISKS WITH THICKNESS VARYING AS $\exp(-\lambda r^q)$

Let

$$h = h_0 \exp(-\lambda r^q) \tag{3.1}$$

and

$$T = \frac{t_0}{\log_e(b/a)} \log_e(b/r)^*. \tag{3.2}$$

Substitution of these values of h and T in equation (1.5) gives

$$r^2 \frac{d^2P}{dr^2} + [3-\lambda q r^q] r \frac{dP}{dr} - [(2+\nu)+(q-1)]\lambda q r^q P = -(3+\nu)\rho r^2\omega^2 + \frac{E\alpha t_0}{\log_e(b/a)}. \tag{3.3}$$

Putting $\zeta = \lambda r^q$ in the equation (3.3) we obtain

$$\zeta \frac{d^2P}{d\zeta^2} + \left(1+\frac{2}{q}-\zeta\right)\frac{dP}{d\zeta} - \frac{1}{q}[1+\nu+q]P$$

$$= -\frac{(3+\nu)\rho\omega^2}{q^2(\lambda)^{2/q}}(\zeta)^{2/q-1} + \frac{E\alpha t_0}{q^2\log_e(b/a)}\cdot\frac{1}{\zeta}. \tag{3.4}$$

(*a*) Let $q = 1$, so that the equation of the profile is

$$h = h_0 e^{-\lambda r}. \tag{3a}$$

For $q = 1$, equation (3.4) reduces to

$$\zeta \frac{d^2P}{d\zeta^2} + (3-\zeta)\frac{dP}{d\zeta} - \sigma P = -k_1\zeta + \frac{k_2}{\zeta} \tag{3a.1}$$

where

$$\sigma = 2+\nu, \quad k_1 = \frac{(3+\nu)\rho\omega^2}{\lambda^2}, \quad \text{and} \quad k_2 = \frac{E\alpha t_0}{\log_e(b/a)}. \tag{3a.2}$$

The complementary function of the equation (3a.1) in terms of the confluent hypergeometric function $M(\sigma, 3, \zeta)$ which stands for the convergent series

$$1 + \frac{\sigma}{3}\zeta + \frac{\sigma(\sigma+1)}{3.4}\frac{\zeta^2}{2!} + \frac{\sigma(\sigma+1)(\sigma+2)}{3.4.5}\frac{\zeta^3}{3!} + \cdots \text{ to infinity}$$

(Webb and Airey, 1918) is

$$[A_1 + B_1\log\zeta]M(\sigma, 3, \zeta) + B_1\left[\frac{\sigma}{3}\zeta\left(\frac{1}{\sigma}-\frac{1}{3}-1\right) + \frac{\sigma(\sigma+1)}{3.4}\frac{\zeta^2}{2!}\left(\frac{1}{\sigma}+\frac{1}{\sigma+1}-\frac{1}{3}-\frac{1}{4}-1-\frac{1}{2}\right)\right.$$

$$\left. + \frac{\sigma(\sigma+1)(\sigma+2)}{3.4.5}\frac{\zeta^3}{3!}\left(\frac{1}{\sigma}+\frac{1}{\sigma+1}+\frac{1}{\sigma+2}-\frac{1}{3}-\frac{1}{4}-\frac{1}{5}-1-\frac{1}{2}-\frac{1}{3}\right) + \cdots \text{ to infinity}\right]$$

---

[*] If instead we take $T = t_b + \frac{t_a - t_b}{\log_e(b/a)}\log_e(b/r)$, we are only to put $t_a - t_b$ for t_0 in our equations.

$$= A_1 M(\sigma, 3, \zeta) + B_1 \phi_1(\zeta) \tag{3a.3}$$

where

$$\phi_1(\zeta) = (\log \zeta) M(\sigma, 3, \zeta) + \left[\frac{\sigma}{3} \zeta \left(\frac{1}{\sigma} - \frac{1}{3} - 1 \right) \right.$$

$$\left. + \frac{\sigma(\sigma+1)}{3.4} \frac{\zeta^2}{2!} \left(\frac{1}{\sigma} + \frac{1}{\sigma+1} - \frac{1}{3} - \frac{1}{4} - 1 - \frac{1}{2} \right) + \cdots \text{to infinity} \right] \tag{3a.4}$$

and A_1, B_1, are the constants of integration to be determined from the boundary conditions. (The tables of this function have been given in the *Reports of the Math. Tables Comm., British Association*, 1927).

As a particular integral of the equation (3a.1), we assume

$$P = A_2 + B_2 \zeta + \frac{C_2}{\zeta} + \frac{D_2}{\zeta^2}. \tag{3a.5}$$

We find that this equation is satisfied when

$$A_2 = \frac{3\rho\omega^2}{\lambda^2(2+\nu)}. \qquad B_2 = \frac{\rho\omega^2}{\lambda^2}$$

$$C_2 = -\frac{E\alpha t_0}{(1+\nu)\log_e(b/a)}, \quad D_2 = \frac{E\alpha t_0}{\nu(1+\nu)\log_e(b/a)}. \tag{3a.6}$$

From (3a.3), (3a.5), and (3a.6) the general solution of (3a.1) is determined as

$$P = A_1 M(\sigma, 3, \zeta) + B_1 \phi_1(\zeta) + \frac{3\rho\omega^2}{\lambda^2(2+\nu)} + \frac{\rho\omega^2}{\lambda^2}\zeta$$

$$- \frac{E\alpha t_0}{(1+\nu)\log_e(b/a)} \cdot \frac{1}{\zeta} + \frac{E\alpha t_0}{\nu(1+\nu)\log_e(b/a)} \cdot \frac{1}{\zeta^2}. \tag{3a.7}$$

From the boundary conditions

$$(P)_{\substack{r=b \\ \text{or } \zeta=\lambda b}} = 0 \qquad (P)_{\substack{r=a \\ \text{or } \zeta=\lambda a}} = 0$$

we have

$$A_1 M(\sigma, 3, \lambda b) + B_1 \phi_1(\lambda b) + L_1 = 0,$$
$$A_1 M(\sigma, 3, \lambda a) + B_1 \phi_1(\lambda a) + L_2 = 0. \tag{3a.8}$$

where

$$L_1 = \frac{3\rho\omega^2}{\lambda^2(2+\nu)} + \frac{\rho\omega^2 b}{\lambda} - \frac{E\alpha t_0}{(1+\nu)\log_e(b/a)} \left[\frac{1}{\lambda b} - \frac{1}{\nu\lambda^2 b^2} \right],$$

$$L_2 = \frac{3\rho\omega^2}{\lambda^2(2+\nu)} + \frac{\rho a \omega^2}{\lambda} - \frac{E\alpha t_0}{(1+\nu)\log_e(b/a)} \left[\frac{1}{\lambda a} - \frac{1}{\nu\lambda^2 a^2} \right]. \tag{3a.9}$$

From the equations (3a.8), we get

$$\frac{A_1}{\begin{vmatrix} \phi_1(\lambda b), & L_1 \\ \phi_1(\lambda a), & L_2 \end{vmatrix}} = \frac{B_1}{\begin{vmatrix} L_1, & M(\sigma, 3, \lambda b) \\ L_2, & M(\sigma, 3, \lambda a) \end{vmatrix}} = \frac{1}{\begin{vmatrix} M(\sigma, 3, \lambda b), & \phi_1(\lambda b) \\ M(\sigma, 3, \lambda a), & \phi_1(\lambda a) \end{vmatrix}} \tag{3a.10}$$

whence A_1 and B_1 are determined.

Using (3a) in the equation (1.2) and noting that $\zeta = \lambda r$, we find

$$Q = (1-\zeta)P + \zeta\frac{dP}{d\zeta} + \frac{\rho\omega^2\zeta^2}{\lambda^2}. \tag{3a.11}$$

Substituting in this from equation (3a.7) and using the results

$$\frac{d}{dx}M(\alpha, \gamma, x) = \frac{\alpha}{\gamma}M(\alpha+1, \gamma+1, x) = \frac{\alpha-\gamma}{\gamma}[M(\alpha, \gamma+1, x)] + M(\alpha, \gamma, x) \tag{3a.12}$$

we can determine the value of Q.

(*b*) Let $q = 2$. So that the equation of the profile is

$$h = h_0 e^{-\lambda r^2}. \tag{3b}$$

Putting $q = 2$ in the equation (3.4), we have

$$\zeta\frac{d^2P}{d\zeta^2} + (2-\zeta)\frac{dP}{d\zeta} - \beta P = -k_3 + k_4\frac{1}{\zeta} \tag{3b.1}$$

where

$$\beta = \frac{3+\nu}{2}, \quad k_3 = \frac{(3+\nu)\rho\omega^2}{4\lambda}, \quad k_4 = \frac{E\alpha t_0}{4\log_e(b/a)}. \tag{3b.2}$$

As a particular integral of the equation (3b.1) we assume

$$P = A_3 + \frac{B_3}{\zeta}. \tag{3b.3}$$

We find that the equation is satisfied, when

$$A_3 = \frac{\rho\omega^2}{2\lambda}, \quad B_3 = -\frac{E\alpha t_0}{2(1+\nu)\log_e(b/a)}. \tag{3b.4}$$

The complementary function of the equation (3b.1) in terms of confluent hypergeometric function $M(\beta, 2, \zeta)$ is

$$\begin{aligned}[C_3 + D_3\log_e\zeta]M(\beta, 2, \zeta) &+ D_3\Big[\tfrac{1}{2}\beta\zeta\left(\frac{1}{\beta} - \frac{1}{2} - 1\right) + \frac{\beta(\beta+1)}{2.3}\frac{\zeta^2}{2!}\left(\frac{1}{\beta} + \frac{1}{\beta+1} - \frac{1}{2} - \frac{1}{3} - 1 - \frac{1}{2}\right) \\ &+ \frac{\beta(\beta+1)(\beta+2)}{2.3.4}\frac{\zeta^3}{3!}\left(\frac{1}{\beta} + \frac{1}{\beta+1} + \frac{1}{\beta+2} - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - 1 - \frac{1}{2} - \frac{1}{3}\right) + \cdots \text{to infinity}\Big] \\ &= C_3M(\beta, 2, \zeta) + D_3\phi_2(\zeta), \end{aligned} \tag{3b.5}$$

where C_3, D_3 are constants of integration to be determined from the boundary conditions

and

$$\phi_2(\zeta) = [\log_e \zeta]\, M(\beta, 2, \zeta) + \Big[\tfrac{1}{2}\beta\zeta\left(\frac{1}{\beta} - \frac{1}{2} - 1\right) + \frac{\beta(\beta+1)}{2.3}\,\frac{\zeta^2}{2!}\left(\frac{1}{\beta} + \frac{1}{\beta+1} - \frac{1}{2} - \frac{1}{3} - 1 - \frac{1}{2}\right) + \cdots \text{to infinity}\Big]. \tag{3b.6}$$

The results (3b.3), (3b.5) and (3b.4) allow us to write the general solution of (3b.1) in the form

$$P = C_3 M(\beta, 2, \zeta) + D_3\phi_2(\zeta) + \frac{\rho\omega^2}{2\lambda} - \frac{E\alpha t_0}{2(1+\nu)\log_e (b/a)}\cdot\frac{1}{\zeta}. \tag{3b.7}$$

From the boundary conditions

$$(P)_{\substack{r=b \\ \text{or } \zeta=\lambda b^2}} = 0 \qquad (P)_{\substack{r=a \\ \text{or } \zeta=\lambda a^2}} = 0$$

we have,

$$\begin{aligned} C_3 M(\beta, 2, \lambda b^2) + D_3\phi_2(\lambda b^2) + L_3 &= 0, \\ C_3 M(\beta, 2, \lambda a^2) + D_3\phi_2(\lambda a^2) + L_4 &= 0. \end{aligned} \tag{3b.8}$$

where

$$L_3 = \frac{\rho\omega^2}{2\lambda} - \frac{E\alpha t_0}{2(1+\nu)\log_e (b/a)}\cdot\frac{1}{\lambda b^2}, \quad \text{and} \quad L_4 = \frac{\rho\omega^2}{2\lambda} - \frac{E\alpha t_0}{2(1+\nu)\log_e (b/a)}\cdot\frac{1}{\lambda a^2}. \tag{3b.9}$$

From the equations (3b.8), we have

$$\frac{C_3}{\begin{vmatrix} \phi_2(\lambda b^2), & L_3 \\ \phi_2(\lambda a^2), & L_4 \end{vmatrix}} = \frac{D_3}{\begin{vmatrix} L_3, & M(\beta, 2, \lambda b^2) \\ L_4, & M(\beta, 2, \lambda a^2) \end{vmatrix}} = \frac{1}{\begin{vmatrix} M(\beta, 2, \lambda b^2), & \phi_2(\lambda b^2) \\ M(\beta, 2, \lambda a^2), & \phi_2(\lambda a^2) \end{vmatrix}}. \tag{3b.10}$$

whence C_3 and D_3 are determined.

Using (3b) in the equation (1.2) and noting that $\zeta = \lambda r^2$, we obtain

$$Q = (1-2\zeta)P + 2\zeta\frac{dP}{d\zeta} + \frac{\rho\omega^2\zeta}{\lambda}. \tag{3b.11}$$

Substituting into this the value of P from the equation (3b.7) and using the results of (3a.12) we can completely determine Q.

THE VALUE OF λ

Let $2h_b$ and $2h_a$ be the thickness at the rim and at the bore respectively, so that

$$h_a/h_b = e^{\lambda(b^2-a^2)}$$

whence,

$$\lambda = \frac{2{\cdot}303}{b^2-a^2}\log_{10}\frac{h_a}{h_b}. \tag{3b.12}$$

NUMERICAL CALCULATIONS

For a particular case, taking $b = 24''$, $a = 6''$, $2h_a = 6''$, $2h_b = 2{\cdot}4''$ and $\nu = 0{\cdot}3$. we have, from (3b.12) $\lambda = 0{\cdot}0017$

$$\lambda b^2 = 0{\cdot}9792,\quad M(\beta,\,2,\,\lambda b^2) = 2{\cdot}2903,\quad \phi_2(\lambda b^2) = -1{\cdot}4149.$$

$$\lambda a^2 = 0{\cdot}0612,\quad M(\beta,\,2,\,\lambda a^2) = 1{\cdot}0519,\quad \phi_2(\lambda a^2) = -1{\cdot}3239$$

$$L_3 = 294{\cdot}1\rho\omega^2 - {\cdot}2833E\alpha t_0,\quad C_3 = 17{\cdot}29\rho\omega^2 - 3{\cdot}912E\alpha t_0$$

$$L_4 = 294{\cdot}1\rho\omega^2 - 4{\cdot}533E\alpha t_0,\quad D_3 = 236\rho\omega^2 - 6{\cdot}5349E\alpha t_0.$$

In conclusion I thank Dr. B. Sen for his kind help in the preparation of this paper.

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ON THE MOMENTS OF NON-CENTRAL χ^2

By

M. C. CHAKRABARTI

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1. Let $x_1, x_2, \ldots, x_n$ be n independent stochastic variates having normal distributions with means $\mu_1, \mu_2, \ldots, \mu_n$ respectively and a common standard deviation σ. The statistic

$$\chi'^2 = \sum_{i=1}^{n} x_i^2/\sigma^2$$

is known as non-central χ^2. The object of this paper is to derive certain recurrence relations among the moments of this statistic. The relations obtained are believed to be new.

2. Without loss of generality, we shall assume $\sigma = 1$, so that

$$p(x_i) = \{1/(2\pi)^{\frac{1}{2}}\}\exp\{-(x_i-\mu_i)^2/2\}, \quad i = 1, 2, \ldots, n.$$

The characteristic function of the distribution law of χ'^2 is

$$\varphi(t) = \exp\{(\lambda it)/(1-2it)\}/(1-2it)^{n/2} \tag{1}$$

$$= \exp[-\tfrac{1}{2}\lambda+\lambda/\{2(1-2it)\}]/(1-2it)^{n/2} \tag{2}$$

where

$$\lambda = \sum_{i=1}^{n} \mu_i^2.$$

Hence the distribution law of χ'^2 is

$$p(\chi'^2) = \frac{\exp(-\frac{1}{2}\lambda)}{2\pi}\int_{-\infty}^{\infty}\exp[-it\chi'^2+\lambda/\{2(1-2it)\}]/(1-2it)^{n/2}dt$$

$$= \frac{\exp(-\frac{1}{2}\lambda)}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-it\chi'^2}}{(1-2it)^{n/2}}\left[\sum_{r=0}^{\infty}\frac{\lambda^r}{2^r.r!\;(1-2it)^r}\right]dt$$

$$= \frac{\exp(-\frac{1}{2}\lambda)}{2\pi}\sum_{r=0}^{\infty}\frac{\lambda^r}{2^r.r!}\int_{-\infty}^{\infty}\frac{e^{-it\chi'^2}}{(1-2it)^{r+n/2}}dt,$$

the termwise integration being obviously permissible. Therefore

$$p(\chi'^2) = \frac{\exp(-\frac{1}{2}\lambda)}{2\pi}\sum_{r=0}^{\infty}\frac{\lambda^r}{2^r.r!}\frac{e^{-\frac{1}{2}\chi'^2}}{2^{r+n/2}}.\frac{2\pi}{\Gamma(r+n/2)}(\chi'^2)^{r+\frac{1}{2}n-1}$$

$$= \frac{\exp\{-\frac{1}{2}(\lambda+\chi'^2)\}}{2^{n/2}}\sum_{r=0}^{\infty}\frac{(\chi'^2)^{r+\frac{1}{2}n-1}.\lambda^r}{2^{2r}.r!\;\Gamma(r+\frac{1}{2}n)} \tag{3}$$

$$= \frac{\exp\{-\frac{1}{2}(\lambda+\chi'^2)\}}{2(i\lambda^{\frac{1}{2}}\chi')^{\frac{1}{2}(n-2)}}(\chi')^{\frac{1}{2}(n-2)}.J_{\frac{1}{2}(n-2)}(i\lambda^{\frac{1}{2}}\chi') \tag{4}$$

where $i = \sqrt{(-1)}$ and $J_{\frac{1}{2}(n-2)}(i\lambda^{\frac{1}{2}}\chi')$ is the Bessel function of order $\frac{1}{2}(n-2)$ and imaginary argument. (3) and (4) have been given by Fisher (1928), Tang (1938), Patnaik (1949) by different methods.

Incidentally (1) gives the rth cumulant k_r as

$$k_r = 2^{r-1}(r-1)!\ (n+r\lambda)$$

whence we get the recurrence relation

$$k_r = 2(n+r\lambda)\frac{dk_{r-1}}{d\lambda}. \tag{5}$$

Thus, knowing $k_1 = n+\lambda$, we get successively from (5)

$$k_2 = 2(n+2\lambda), \quad k_3 = 8(n+3\lambda), \quad k_4 = 48(n+4\lambda)$$

and so on.

3. From (4), we get

$$\mu_k = \frac{\exp(-\frac{1}{2}\lambda)}{(i\lambda^{\frac{1}{2}})^{\frac{1}{2}(n-2)}}\int_0^\infty (\chi')^{2k+\frac{1}{2}n}\exp(-\tfrac{1}{2}\chi'^2)J_{\frac{1}{2}(n-2)}(i\lambda^{\frac{1}{2}}\chi')d\chi'$$

$$= \frac{2^k\Gamma(k+\frac{1}{2}n)}{\Gamma(\frac{1}{2}n)}\cdot {}_1F_1(-k,\ \tfrac{1}{2}n,\ -\tfrac{1}{2}\lambda) \tag{6}$$

(*Cf.* formula (8), p. 394 of Watson's Bessel Function), where ${}_1F_1$ is the usual generalised hypergeometric function. The result (6) can also be obtained from (3) if we multiply both sides by $(\chi')^{2k}$, integrate term by term and make use of the well-known relation

$${}_1F_1(a,\ b,\ x) = e^x\cdot{}_1F_1(b-a,\ b,\ -x)$$

due to Kummer.

From (6), we can easily deduce

$$\mu'_k = \sum_{r=0}^{\infty}\binom{k}{r}\lambda^r(n+2r)(n+2r+2)\ldots(n+2k-2) \tag{7}$$

$$= 2^k\left[\frac{d^k}{dx^k}\{e^{\frac{1}{2}\lambda x}(1+x)^{\frac{1}{2}n+k-1}\}\right]_{x=0} \tag{8}$$

whence we get the recurrence relation

$$\mu'_{k+1} = \frac{\lambda}{k+1}\cdot\frac{d\mu'_{k+1}}{d\lambda}+(n+2k)\mu'_k \tag{9}$$

Hence

$$\mu'_k = \frac{1}{n+2k}\mu'_{k+1}-\frac{\lambda}{(k+1)(n+2k)}\frac{d\mu'_{k+1}}{d\lambda}, \quad k = 0,\ 1,\ 2,\ldots \tag{10}$$

$$\mu'_{k+1} = \lambda^{k+1}+(n+2k)(k+1)\lambda^{k+1}\int_\lambda^\infty \frac{\mu'_k}{\lambda^{k+2}}\,d\lambda, \quad k = 0,\ 1,\ 2,\ldots \tag{11}$$

(10) can be used for calculating $\mu'_k, \mu'_{k-1}, \ldots$ from μ'_{k+1} and (11) can be used for the calculation of $\mu'_{k+1}, \mu'_{k+2}, \ldots$ from μ'_k.

4. The kth moment about the mean

$$\mu_k = \sum_{r=0}^{k} (-1)^r \binom{k}{r} \mu'_{k-r} (n+\lambda)^r$$

whence, we get easily

$$\frac{d\mu_k}{d\lambda} = \frac{k}{\lambda}\mu_k - \frac{2k(k-1)}{\lambda}[\mu_{k-1} + (n+\lambda)\mu_{k-2}].$$

Therefore

$$\mu_k = 2k(k-1)\lambda^k \int_{\lambda}^{\infty} \frac{\mu_{k-1} + (n+\lambda)\mu_{k-2}}{\lambda^{k+1}} d\lambda, \quad k = 2, 3, 4, \ldots \tag{12}$$

Since $\mu_0 = 1$, $\mu_1 = 0$, we get successively

$$\mu_2 = 2(n+2\lambda), \quad \mu_3 = 8(n+3\lambda), \quad \mu_4 = 12(n+2\lambda)^2 + 48(n+4\lambda);$$

$$\beta_1 = \frac{8(n+3\lambda)^2}{(n+2\lambda)^3}, \quad \beta_2 = 3 + \frac{12(n+4\lambda)}{n+2\lambda}.$$

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QUASI-GROUPS AND NONASSOCIATIVE SYSTEMS—II

By

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1. Introduction

In a group, there exists a unique element which is the common identity for every element of the group. Thus if e is this element, then $ea = ae = a$ for every element a of the group. But it is wellknown that the case is otherwise in a quasigroup. Here each element has distinct left and right identity elements which are again different for different elements of the quasigroup. If every element of a quasigroup has the same left (or right) identity element, then the quasigroup is called a left (or right) loop. If a quasigroup is both a left and a right loop, then it is called a loop. Of course, every quasigroup can be distorted homotopically into a loop, but during this distortion, many important properties are lost. Thus the loop obtained by distortion is distinct from the original quasigroup in many respects. Though a homomorphism is a special case of a homotopy, yet it is simple enough to preserve,many important properties of a quasigroup. One such important property is the identity relations of the different elements.

In the previous paper (Choudhury, 1948) having the same title, it was shown that an inclusion relation can be defined in a commutative quasigroup by means of the identity relation. But the commutativity condition used there is not essential. Indeed, a right inclusion and a left inclusion can be defined by means of the right identity element and the left identity element respectively. With the help of each of these inclusion relations, it is possible to draw a graph, a right graph or a left graph as the case may be, in a manner similar to the familiar Hasse diagram of a partially ordered system. Of course, the graph of a quasigroup differs in many respects from that of a partially ordered system. The inclusion relation here is not transitive and hence an element includes only the element lying just behind it and not those elements which lie far behind it. Further the graph of a quasigroup contains cycles of elements while the graph of a partially ordered system can not have such things.

The graph of a loop is the simplest one It is a star shaped figure in which the common identity element lie in the centre and the other elements of the loop are connected to it. The weight of the identity element is $n-1$ if n is the number of elements in a finite loop ; but all other elements are source elements. The graph of a finite quasigroup consists of a finite number chains with cycles. The graph of certain infinite guasigroups consists of infinite chains only.

The graphs of a quasigroup are important for the study of the homomorphisms of the quasigroup. If $ae = a$ and α is a homomorphism of the quasigroup, then $\alpha(a)\alpha(e) = \alpha(a)$, *i.e.*, if e is the right identity of a, then the corresponding element $\alpha(e)$ is also the right identity of $\alpha(a)$, unless $\alpha(a) = \alpha(e)$ in which $\alpha(e)$ is idempotent. Therefore, if one element lie just in front of another element in the graph of a quasigroup, then their corresponding elements will be similarly situated in the graph of the homomorphic quasigroup, unless these corresponding elements become identical and lie at the end of a chain. Further the weight $w(a)$ of an element a is the number of solutions of the equations $xa = x$. If x_i are the solutions, then $x_i a = x_i$ and hence $\alpha(x_i)\alpha(a) = \alpha(x_i)$, $i = 1, 2, \ldots, w(a)$. Thus $\alpha(x_i)$ are solutions of the equation $y\alpha(a) = y$. But some of $\alpha(x_i)$ may be equal to $\alpha(x_k)$ and hence the number of solutions of the equation $y\alpha(a) = y$ is less than $w(a)$. Hence $w(\alpha(a))$ is less than or is equal to $w(a)$. Therefore if a is a source element, then $\alpha(a)$ is also a source element. All these results show the important connections between the graph of the quasigroup and the homomorphism.

The structure of the quasigroup in relation to the identity elements becomes clear if the I-sets and H-sets are studied. An I-set is a subset of the quasigroup such that if a is an element of I-set, then both the left and the right identity of a belong to it. The I-sets determine a topology in the quasigroup. Similarly H-sets are subsets such that if a is an element of it, then all elements which has a either as a left identity or a right identity belong to it. These H-sets also determine another topology in the quasigroup. These two topologies determine the character of the quasigroup.

2. Chain of identity elements

In a quasigroup Q, it is possible to determine the chain of right identity elements

$$e_0 \leq e_1 \leq e_2 \leq \cdots \leq e_n \leq e_{n+1} \leq \cdots$$

where e_{n+1} is the right identity of e_n, if the first element e_0 is given. Here, either every element of the chain are distinct in which case the chain must be infinite, or all elements of the chain are not distinct in which case a certain element e_m is identical with a certain element e_n, $m > n \geq p$. As the right identity element of every element is uniquely determined, it follows from the fact ~~that~~ $e_m = e_n$, that $e_{m+1} = e_{n+1}$ and therefore by mathematical induction $e_{m+i} = e_{n+i}$. Thus the chain reduces to

$$e_0 \leq e_1 \leq e_2 \leq \cdots \leq e_p \leq e_{p+1} \leq \cdots \leq e_{p+r} = e_p.$$

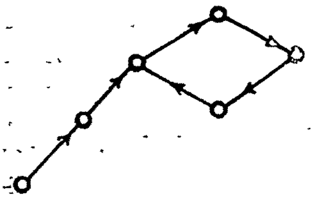
The chain is therefore finite. The elements $e_p, e_{p+1}, \ldots, e_{p+r-1}$ formed a closed chain and are said to form a cycle of length r. The graph of the chain is given in the figure. If the length r is equal to 1, then e_p is idempotent. Further in an Abelian quasigroup, the length r is not equal to 2. For if $r = 2$ in such a case,

$$e_p \leq e_{p+1} \leq e_p.$$

Thus

$$e_p . e_{p+1} = e_p \quad \text{and} \quad e_{p+1} . e_p = e_{p+1}.$$

Hence, as the quasigroup is commutative, $e_p = e_{p+1}$ and e_p is therefore idempotent.



The chain may also be continued behind e_0

$$e_{-n} \leq e_{-n+1} \leq \cdots \leq e_{-1} \leq e_0 \leq e_1 \leq \cdots$$

until an element e_{-n} is obtained which is a source element. It may also happen that the source element is never arrived at and the chain may be extended behind e_0 indefinitely.

If a source element is obtained, the chain is finite also behind and the chain can then be renumbered in such a manner that e_0 is the source element. In this case, e_0 will be called the generator of the chain.

An element e can not have a cycle behind it unless e belongs to the cycle. For, if e does not belong to a cycle, continue the chain behind e. If a cycle is obtained in this process, then an element is obtained having two right identity elements. This is impossible. This result proves that if a chain has no generator, it is infinite on the back.

It will be interesting to consider now some examples. The set of positive integers when distorted by the law

$$a^*b = a-b$$

becomes a quasigroup and in this quasigroup, the left identity of an element a is the solution of the equation $x^*a = a$. In terms of ordinary subtraction, this is equivalent to $x-a = a$ and hence $x = 2a$. The element a generates a chain

$$a \leq 2a \leq 4a \leq \cdots \leq 2^n a \leq \cdots.$$

Each integer a generates therefore an infinite chain in front of it; but it is otherwise at the back of it. If $a = 2^m.p$, $(p, 2) = 1$, then the chain can be continued behind it upto m elements

$$p = \frac{a}{2^m} \leq \frac{a}{2^{m-1}} \leq \cdots \leq \frac{a}{2} \leq a \leq 2a \leq 2^2 a \leq \cdots$$

Thus every odd integer is a source element and all the integers are distributed in these chains in such a manner that any given integer will belong to one and only one such chain.

All the elements of a quasigroup may lie in one chain. For example, the four elements a, b, c, d of the quasigroup whose multiplication table is the following

	a	b	c	d
a	d	a	c	b
b	c	d	b	a
c	a	b	d	c
d	b	c	a	d

lie in one chain of right identities

$$a \leq b \leq c \leq d.$$

A finite quasigroup has only a finite number of source elements generating only a finite number of chains. The best example of this is obtained from the consideration of the cycles generated by an elements a in the quasigroup $GQ_{p^n}(k, s)$. The multiplication in $GQ_{p^n}(k, s)$ is defined by the rule $a^*b = a^k b^s$ and here k, s are both prime to p^n-1. The left identity of a is the solution of the equation $x^*a = a$ in $GQ_{p^n}(k, s)$. But this solution is the solution of the equation $x^k a^s = a$ in GF_{p^n}. As k is prime to p^n-1, this equation is solvable and the solution is $x = a^{\frac{1-s}{k}}$. Hence if $\frac{1-s}{k} = r$, the chain of left identities is

$$a \leq a^r \leq a^{r^2} \leq a^{r^3} \leq \cdots \leq a^{r^n} \leq \cdots.$$

As $GQ_{p^n}(k, s)$ is a finite system, the chain must be finite and therefore must have cycles. Different cases occur here. If $r = 1$, then $s+k = 1$ and a is idempotent. Thus if a Galois field is distorted by the rule $a*b = a^{1-s}b^s$, then every element of $GQ_{p^n}(k, s)$ so obtained is idempotent. If r is a positive integer greater than 1, then the chain will consist of some powers of a. Suppose that the order of the element a in the multiplicative group of GF_{p^n} is m. Then

$$r^n = lm + t \quad i.e., \quad r^n \equiv t \mod m.$$

If $t = 0$ for any r^n, then $a^{r^n} = 1$ and the chain ends there as 1 is idempotent. But if $t \neq 0$ for any power of r, then the chain will consist of those powers a^t of a for which t is a power of the class (r) in the ring of the classes of residues mod m. The integer r is fixed by the distortion only while the integer m is fixed by the element a. For example, if $r = 3$ and $m = 5$, then the cycle is $a \leq a^3 \leq a^4 \leq a^2 \leq a$; but if $r = 3$ and $m = 11$, then the cycle is $a \leq a^3 \leq a^9 \leq a^5 \leq a^4 \leq a$. It is evident from the examples that a cycle will not, in general, include all powers of a and therefore it will be interesting to consider the cycles generated by different powers of a. If $b = a^l$ and m is the order of a, then the order of b is $\lambda = m/(l, m)$ where (l, m) is the *h.c.f* of l and m. The cycle generated by b is given by

$$b \leq b^r \leq b^{r^2} \leq b^{r^3} \leq \cdots \leq b^{r^n} \leq \cdots.$$

$r^n \equiv t \mod \lambda$ and $b^t \equiv a^{lt} \equiv a^\mu$ where $\mu \equiv lt \mod m$. Now $m = \lambda(l, m)$ and let $l = \lambda'(l, m)$; then $l\lambda = \lambda' m$. Also as $r^n \equiv t \mod \lambda$, and $lt \equiv \mu \mod m$

$$r^n = \lambda\nu + t \text{ and } lt = \theta m + \mu.$$

Hence

$$lr^n = \lambda\nu l + lt = \lambda\nu l + \theta m + \mu = \nu\lambda' m + \theta m + \mu = (\nu\lambda' + \theta)m + \mu \equiv \mu \mod m.$$

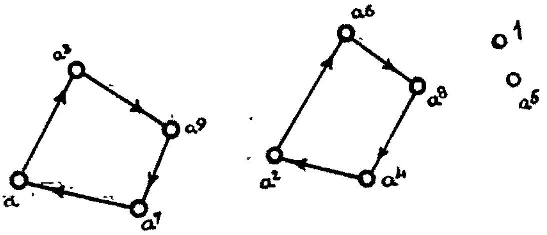
Thus if $l = r^k$, then $lr^n = r^{n+k}$. Therefore if b occurs in the chain generated by a, the chain generated by $b = a^l$ is a subchain of that generated by a. As $\mu = lt \mod m$, the powers of a in the chain generated by a^l is obtained, in case $l \neq r^k$, from those of the chain generated by a by multiplying with $l \mod m$.

If now $(r, m) = 1$, then $\lambda_1 r + \lambda_2 m = 1$, *i.e.*, $\lambda_1 r \equiv 1 \mod m$. Raising both sides to nth power,

$$\lambda_1^n r^n \equiv 1 \mod m.$$

Thus $r^{-n} \mod m$ exists and r^n is not a divisor of zero in the ring of the classes of residues mod m. Hence if $r^n \equiv t \mod m$, $(t, m) = 1$, *i.e.*, every power of r is prime to m. If $O(a^t)$ is the order of the element a^t in the multiplicative semigroup of the ring of the classes of residues, it is evident that $O(a^t) = m/(t, m)$, where (t, m) is the *h.c.f.* of t and m. As $(t, m) = 1$, $O(a^t) = m$. Thus if r is prime to m, all elements belonging to the chain generated by a have the same order m as a. As any other element belonging to the chain of a will generate a subchain, one may call the chain of a as a chain of order m.

Next consider the chain generated by a^l; $l \neq r^k$. In this case, $lr^n \neq r^{n+k}$ and hence a^l will generate a new chain. The elements belonging to this cycle will be a^μ where $\mu \equiv lt \mod m$. As $a^\mu = a^{lt} = (a^t)^l = c^l$ where $c = a^t$; $O(c) = m$ and hence $O(c^l) = m/(l, m)$. Thus $O(a^\mu) = m/(l, m)$ and therefore the order of the chain generated by a^l is $m/(l, m)$. If $(r, m) = 1$ and $O(a) = m$, then the powers of a in GF_{p^*} are distributed in chains whose orders are divisors of m. For example, if $r = 3$ and $m = 10$, the powers a^l are distributed throughout the graph. Of course, there might be more than one cycles of same order among which the powers of a are distributed. For example, if $m = 1$, $r = 3$, the powers of a are distributed in the cycles:



$$a \leq a^3 \leq a^9 \leq a^5 \leq a^4 \leq a,$$

$$a^2 \leq a^6 \leq a^7 \leq a^{10} \leq a^8 \leq a^2,$$

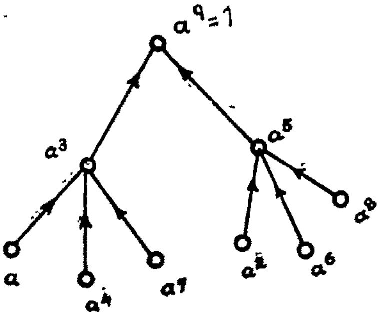
$$a^{11} \leq a^{11}.$$

The first two cycles have the same order namely 11.

But the situation is greatly different if the $h.c.f.\ (r, m) = s \neq 1$. Here $r = sv_1$; $m = sv_2$ and hence

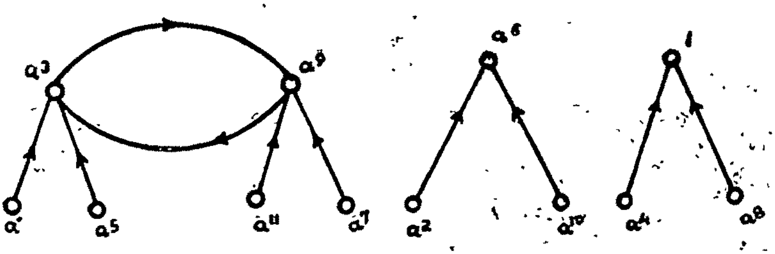
$$r^n v_2 = sv_2 . s^{n-1} v_1{}^n = ms^{n-1} v_1{}^n \equiv 0 \mod m.$$

Thus r^n is a divisor of zero for all n. In particular, one has to consider two distinct cases: (i) $r^n = m$ i.e., m is a certain power of r and (ii) r and m have a common divisor. As examples, one can construct the graphs when (i) $r = 3$, $m = 3^2$, (ii) $r = 3$, $m = 12$.



(i)

In above, only the simplest type of distortion $a*b = a^k b^s$ where $(1-s)/k = r$ is an integer is considered. If r is an arbitrary rational number, the case becomes more complicated as one has to extract a root here. Hence this case will not be discussed here.



(ii)

3. *I*-SETS, *H*-SETS.

The set of elements forming a chain has the property that if a is any element of the set, then the right identity of a will also belong to the set. It is now proposed to generalise this idea. Let X be a subset of the quasigroup Q and $\mu(X)$ be that subset of Q which is formed by the right identities of the elements of X. Let us now make convention that $\mu^0(X) = X$ and $\mu^{n+1}(X)$ is the set of right identities of the elements of $\mu^n(X)$, that is

$$\mu^{n+1}(X) = \mu(\mu^n(X)).$$

The join $\bar{X}$ of all subsets $\mu^n(X)$, $n = 0, 1, 2, 3, \ldots$ is a subset of Q having important properties. If a is an element of $\bar{X}$, the right identity of a must belong to $\bar{X}$. On account of this property, the set $\bar{X}$ will be called a right identity closed set or more simply, an *RI*-set. In particular, if X consists of only one element a, then $\bar{X}$ will be the set forming the chain generated by a. Similarly it is possible to consider the left identities of the elements of X and to get the left identity closed sets X^* or more simply, *LI*-sets X^*, which will be characterised by the property that if a is an element of X^*, then the left identity of a will belong to X^*. If Q is a commutative quasigroup, then all *RI*-sets $\bar{X}$ are also *LI*-sets X^*; but otherwise, only certain *RI*-sets will also be *LI*-sets and these will be called *I*-sets. A quasiqroup Q has at least one *I*-set, namely Q itself. At present, the discussion will be restricted to *RI*-sets only.

The subset $\bar{X}$ has two important properties:

$$(C\,1) \qquad \bar{X} \supseteq X$$

$$(C\,2) \qquad X \subseteq \bar{Y} \text{ imply } \bar{X} \subseteq \bar{Y}.$$

The first is evident as $\bar{X}$ is the join of X with other sets $\mu^i(X)$. To prove the second, one notes that $\mu^i(\bar{Y}) = \bar{Y}$. Thus $\mu^i(X) \subseteq \mu^i(\bar{Y}) = \bar{Y}$. As $\bar{X} = X \cup \mu(X) \cup \mu^2(X) \cup \cdots$, $\bar{X} \subseteq \bar{Y}$.

It is now necessary to state certain well-known results (Ore, 1944; Everett, 1944). The correspondence $a \rightarrow \bar{a}$ in a lattice L, by which each element a is associated to another element $\bar{a}$ in such a manner that

$$(C\,1) \qquad \bar{a} \geq a$$

$$(C\,2) \qquad a \leq \bar{b} \text{ imply } \bar{a} \leq \bar{b}$$

is called a Galois closure relation in L. The name Galois is attached to this type of closure operations owing to the fact that this type of closure operations can be obtained from a Galois connexion. If $\bar{a} = a$, then a is called a closed element and it can be easily proved that $\bar{\bar{a}} = \bar{a}$ which shows that $\bar{a}$ is a closed element. The closed elements of L form a sublattice L_1 such that the multiplication in L_1 is same as that in L, but the addition in L_1 is different from that in L. The sum of two closed elements a_1 and a_2 in

L_1 is the least closed element in L_1 containing both a_1 and a_2. If, besides (C 1) and (C 2), a further condition hold:

$$(C\ 3)\quad \overline{a+b} = \bar{a}+\bar{b},$$

then the lattice L will be called a general topological space. The lattice L_1 is in this case a proper sublattice and both addition and multiplication in L_1 are same as those in L.

The subsets of a quasigroup Q form a Boolean Algebra in which the sum is the join of two subsets and the product is the meet of two subsets. The correspondence $X \to \bar{X}$ has been proved above to be a Galois closure in this lattice L. The closed sets form a sublattice L_1 of L. For a closed subset X, $\bar{X} = X$ and hence the neccessary and sufficient condition that X may be closed is that

$$\mu(X) \subseteq X.$$

Now if $\mu(X) \subseteq X$ and $\mu(Y) \subseteq Y$, then $\mu(X \cap Y) \subseteq X$ and Y and hence $\mu(X \cap Y) \subseteq X \cap Y$. Thus if X and Y are closed, then $X \cap Y$ is also closed. Therefore the product in the lattice L_1 is also the meet of two closed sets. More generally, let X and Y be two arbitrary subsets of Q, (i.e., not necessarily closed). If a is an element of $X \cap Y$, then a belongs both to X and Y and hence the right identity of a belongs both to $\mu(X)$ and to $\mu(Y)$ and therefore to $\mu(X) \cap \mu(Y)$. Thus $\mu(X \cap Y) \subseteq \mu(X) \cap \mu(Y)$. On the otherhand, let e may be an element of X. Then one has to consider two cases: (*i*) the weight $w(e)$ is equal to 1 and (*ii*) $w(a)$ is greater than 1. In the first case, e is the right identity of only one element a which must therefore belong to both X and Y and hence to $X \cap Y$. Thus $e \in \mu(X \cap Y)$ and hence in this case, $\mu(X \cap Y) \supseteq \mu(X) \cap \mu(Y)$. But in the second case, e is the right identity of more than one elements some of which may belong to X and some to Y. Thus none of these elements may belong to $X \cap Y$ and $\mu(X \cap Y)$ will not contain e. Thus in general,

$$\mu(X \cap Y) \subseteq \mu(X) \cap \mu(Y)$$

where the equality will hold if the weight of every element of $\mu(X \cap Y)$ is 1. By mathematical induction, this leads to $\mu^n(X \cap Y) \subseteq \mu^n(X) \cap \mu^n(Y)$ and hence to $\overline{X \cap Y} \subseteq \bar{X} \cap \bar{Y}$. The closure operation $X \to \bar{X}$ satisfies not only $C1$ and $C2$ but also $C3$; hence it converts L into a topological space. For, let a be an element of the subset $X \cup Y$; then a must belong to either of two subsets X and Y. Hence the right identity of a will belong to either of the two subsets $\mu(X)$ and $\mu(Y)$ and hence must belong to the join $\mu(X) \cup \mu(Y)$. Thus

$$\mu(X \cup Y) \subseteq \mu(X) \cup \mu(Y).$$

Conversely if e is an element of the join of $\mu(X)$ and $\mu(Y)$, then e must belong to either of them. Therefore e must be the right identity of some elements which must exist either in X or in Y and hence in $X \cup Y$. Thus finally e must belong to $\mu(X \cup Y)$. As e is an arbitrary element of $\mu(X) \cup \mu(Y)$,

$$\mu(X) \cup \mu(Y) \subseteq \mu(X \cup Y).$$

Combining the two results,

$$\mu(X) \cup \mu(Y) = \mu(X \cup Y).$$

If X and Y are RI-sets, *i.e.*, closed, then $\mu(X) \subseteq X$ and $\mu(Y) \subseteq Y$. Therefore in this case $\mu(X \cup Y) = \mu(X) \cup \mu(Y) \subseteq X \cup Y$ and hence $X \cup Y$ is also closed. Thus $\overline{X \cup Y} = \bar{X} \cup \bar{Y}$ which proves that $C3$ is true.

The lattice of RI-sets form a proper sublattice of L and as L is a Boolean Algebra, it is distributive. The set $\bar{X}$ can be easily recognised graphically. Indeed, if the graph of Q is drawn and X is any subset of it, then $\bar{X}$ is the set of all elements of X and all elements of Q that lie in front of the elements of X. It is for this reason, the topology introduced by this closure will be called right fore-topology.

Exactly in a similar manner, it can be shown that the correspondence $X \rightarrow X^*$ introduces into L another topology which may be called left fore-topology. The closed sets are in this case LI sets and these closed sets form a sublattice L_2 of L. The meet of these two sublattices L_1 and L_2 will be a new sublattice L_{12} whose elements are I-sets.

A second pair of topologies can also be defined in L. If $\nu(X)$ be the set of those elements of Q of which the elements of X are right identities, then the elements of $\nu(X)$ are obtained as the solutions of the equations $xe = x$, $e \in X$. The solution of this equation exists if $w(e) \geq 1$ and does not exist if $w(e) = 0$. Thus in the graph of right identities of Q, $\nu(X)$ is the set of those elements which lie just behind the elements of X. Then the elements of the set $X' = X \cup \nu(X) \cup \nu^2(X) \cup \cdots$ are therefore the elements of X and those elements of the graph which lie behind the elements of X right up to the source elements. The correspondence $X \rightarrow X'$ is a closure operation satisfying $C1$, $C2$, $C3$. It is enough to verify $C3$. If a is an element of $\nu(X \cup Y)$ and e is the right identity of a, then e belongs to $X \cup Y$ and hence to X or Y. Thus a belongs to $\nu(X)$ or $\nu(Y)$ and hence to $\nu(X) \cup \nu(Y)$. So

$$\nu(X \cup Y) \subseteq \nu(X) \cup \nu(Y).$$

Conversely if a belongs to $\nu(X) \cup \nu(Y)$, then a belongs either to $\nu(X)$ or to $\nu(Y)$. Hence e belongs either to X or to Y and therefore to $X \cup Y$. Thus a belongs to $\nu(X \cup Y)$. As a is an arbitrary element of $\nu(X) \cup \nu(Y)$,

$$\nu(X \cup Y) \supseteq \nu(X) \cup \nu(Y).$$

Thus finally $\nu(X \cup Y) = \nu(X) \cup \nu(Y)$ from which $C3$ can be verified at once.

The closure $X \rightarrow X'$ defines therefore into the lattice L, a topology which shall be called right hind-topology. The closed elements of this topology will form a sublattice M_1 of L. Similarly with the help of the graph of left identities of Q, a left hind-topology can be defined into L whose closed sets will form another sublattice M_2 of L. The elements of M_1 and M_2 will be called RH-sets and LH-sets respectively. The elements of the lattice $M_1 \cap M_2 = M_{12}$ will be called H-sets.

An RI-sets which is also an RH-set will be called RHI-sets. Similarly LHI-sets and HI-sets are defined. These sets are important. For example, in a graph of right identity, the full chain generated by a source element is an *RHI*-sets. These *RHI*-sets divide the elements of Q into classes. Indeed, if two elements a and b are considered equivalent when a lie behind or in front of b, then this equivalence is reflexive, symmetric and transitive. Hence this equivalence will divide the elements of Q into disjoint classes which are *RHI*-sets and the quasigroup Q is the join of these classes.

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Choudhury, A. C., (1948), *Bull. Cal. Math. Soc.*, **40**, 183.
Everett, C. J., (1944), *Trans. Amer. Math Soc*, **56**, 514
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CORRECTIONS TO MY PAPER ON 'AN EXTENSION OF HAMY'S THEOREM, ETC.'

BY

N. L. GHOSH

In the paper with the above title published in the *Bulletin of the Calcutta Mathematical Society*, **41**, 92, 1949, while discussing the limiting case of the sub-case (*a*) in Art. 2 (p. 99), it has been concluded that "the spherical distribution of density of the nature (2.21) is not possible under any gravitational field whatsoever." This statement is not strictly correct.

Equation (2.28) should be corrected by the addition of a constant, in general, so that the difference of the two potential functions Φ and V should have the form

$$V' \equiv \frac{1}{\pi\rho_0 G}.(\Phi - V) = \frac{2}{15}(2z^2 - r^2) + C_1, \tag{1}$$

where C_1 is a constant, instead of (2.25) The previous argument for rejecting the potential (2.25) as being applicable to a real body is no longer valid. In fact (1) is the tidal potential, *to the first approximation*, due to a distant tide-generating body apart from a term denoting the acceleration with which the attracted body falls towards the primary. Hence the correct conclusion should be that the spherical distribution of nature (2.21) of uniformly rotating mass of gas as in (2.2), is possible under the influence of a distant tide-generating body, placed on the axis of rotation.

It is possible to determine the mass m and the distance c of the tide-generating body from the centre of the rotating gas-sphere, by comparing (1) with the usual expression for the tidal potential (to the first approximation) due to the distant body (Jeans, *Problems of Cosmogony and Stellar Dynamics*, 1919, p. 43). We have in fact, the following equations for m and c:

$$\frac{2\pi\rho_0}{15} = \frac{m}{2c^3}; \quad \pi\rho_0 C_1 = \frac{m}{c}.$$

P. Dive arrived at the same conclusion as in the paper cited above (*Rotations Internes des Astres Fluides*, 1980), by a method entirely different. The author regrets that at the time of publishing his paper he had overlooked Dive's results and hence did not make any reference to his work.

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CORRECTIONS TO MY PAPER ON 'A STUDY OF THE GENERALISED LAPLACE INTEGRAL'

By

S. K. Bose

The following errors have unfortunately crept into my paper entitled 'A study of the Generalised Laplace integral' published in *Bull. Cal. Math. Soc.*, **41**, 1949:

Part I. *Bull. Cal. Math. Soc.*, **41**, 9.

Page 10, 10th line from the top, insert $l!$ on the right hand side; 5th line from the bottom, for $\left(-p\frac{d}{dx}\right)^n$ read $\left(-p\frac{d}{dp}\right)^n$.

Page 11, 12th line from the top, for $\frac{\Gamma(9/4+m)\Gamma(9/4-m)}{4p\Gamma(11/4-m)}$ read $\frac{\Gamma(9/4+m)\Gamma(9/4-m)}{4p(11/4-k)}$.

Page 12, 1st line from the top,

$$\text{for } {}_2F_1\left\{\begin{matrix} l+m+\frac{1}{2},\ l-m+\frac{1}{2} \\ l-k+\frac{1}{2} \end{matrix};\ -\alpha^2\right\} \text{ read } {}_2F_1\left\{\begin{matrix} l+m+\frac{1}{2}\ \ l-m+\frac{1}{2} \\ l-k+1 \end{matrix};\ -\alpha^2\right\}.$$

Page 13, in the integral (c) replace x by z and number the next Whittaker-transform of $x^\lambda e^{-\frac{1}{2}x^2}$ as (4).

Page 14, number the result of Example 6 as (7) and in Example 7 for

$${}_2F_1\left\{\begin{matrix} n+\mu+m+2r+5/4,\ n+\mu-m+2r+5/4 \\ n+\mu-k+2r+5/4 \end{matrix};\ \frac{1}{2}-\frac{a}{2p}\right\}$$

read

$${}_2F_1\left\{\begin{matrix} n+\mu+m+2r+5/4,\ n+\mu-m+2r+5/4 \\ n+\mu-k+2r+7/4 \end{matrix};\ \frac{1}{2}-\frac{a}{2p}\right\};$$

1st line from the bottom, for $\mathbf{R}(p) = 0$ read $\mathbf{R}(p) > 0$.

Page 17, in Example 14, for $W_{-\nu,\lambda}(x)$ read $W_{-\nu,\lambda}(\frac{1}{2}x)$ and number this result as (18) instead of (11).

Page 21, 5th line from the bottom, for $\mathbf{R}(m+n-2l+1) < 0$ read $\mathbf{R}(m+n-2l+1) > 0$.

Page 22, 6th line from the top, for $W_{\frac{1}{2}\nu-3/8,\frac{1}{2}m}(\frac{1}{2}x^2)$ read $W_{-\frac{1}{2}\nu-3/8,\frac{1}{2}m}(\frac{1}{2}x^2)$.

Page 25, 7th line from the top, insert $e^{\frac{1}{2}p}$.

Part II. *Bull. Cal. Math. Soc.*, **41**, 59.

Page 61, 17th line from the top, for K- read L-; and between the 1st and 2nd lines from the bottom insert "**1.6.** *Particular cases of our theorem.*"

Page 62, 13th line from the top, for (B, Eqn. 1.2.8) read (B, Eqn. 1.2. 18).

Page 63, to the right hand side of the formula in the 8th line from the top add

$$+\frac{\Gamma(-2m+1)\Gamma(2m)}{\Gamma(-m+\frac{1}{2}-k)\,\Gamma(m+\frac{1}{2}-k)}\sum_{r=0}^{\infty}\frac{\Gamma(r-m+\frac{1}{2}-k)}{r!\,\Gamma(r-2m+1)}(2p)^{r-m+\frac{1}{4}}$$

$$\times\Gamma(\tfrac{1}{2}\lambda-m+r+\tfrac{3}{4})e^{\frac{1}{2}p}\,p^{-\frac{1}{2}(\frac{1}{2}\lambda-m-n+r+5/4)}\,W_{\left\{\begin{smallmatrix}-\frac{1}{2}(\frac{1}{2}\lambda-m+n+r+\frac{1}{4})\\ \frac{1}{2}(\frac{1}{2}\lambda-m-n+r+\frac{1}{4})\end{smallmatrix}\right.}(p)\,; \qquad (1,$$

13th line from the top, for $\mathbf{R}(\frac{1}{2}\lambda\pm m+\frac{1}{4})>0$ read $\mathbf{R}(\frac{1}{2}\lambda\pm m+\frac{3}{4})>0$.

Page 64, 7th line from the top, for $\Gamma(\lambda+2n+n_1+2r+\frac{1}{2})$ read $\Gamma(\lambda+n+n_1+2r+\frac{1}{2})$; and 8th line from the bottom, number the integral as (1).

PART III. *Bull. Cal. Math. Soc.*, **41**, 68.

Page 68, 7th line from the top, for 'relf-reciprocal' read 'self-reciprocal.'

Page 72, 7th line from the top, for $\mathbf{R}(\lambda+\mu_1+\nu+\frac{3}{2})>0$ read $\mathbf{R}(\nu)>-1$.

Page 73, 7th line from the top, for $\varphi(p)$ read $\frac{1}{p}\varphi(p)$.

LUCKNOW UNIVERSITY,
LUCKNOW, INDIA.

CORRECTIONS TO MY PAPER ON 'CERTAIN SELF-RECIPROCAL FUNCTIONS'

By

S. C. MITRA

The following errors have unfortunately crept into my paper entitled 'On certain self-reciprocal functions' published in *Bull. Cal. Math. Soc.*, **41**, 1 (1949):

Page 3, 8th line from the bottom, for $I_0(x)K_0(x)$ read $I_0(\frac{1}{2}x)K_0(\frac{1}{2}x)$.

Page 3, 5th line from the bottom, and page 4, 5th line from the top, for x please read $\frac{1}{2}x$ everywhere in the lines.

Page 3, last but one line and last line, for $I_0(\frac{1}{2}\sqrt{\pi}.x)K_0(\frac{1}{2}\sqrt{\pi}.x)$ please read $I_0(\frac{1}{4}\sqrt{\pi}.x)K_0(\frac{1}{4}\sqrt{\pi}.x)-\cdots$.

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